

MODULAR FORMS AND PERIOD POLYNOMIALS

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ABSTRACT. We study the space of period polynomials associated with modular forms of integral weight for finite index subgroups of the modular group. For the full modular group, this space is endowed with a pairing, corresponding to the Petersson inner product on modular forms via a formula of Haberland, and with an action of Hecke operators, defined algebraically by Zagier. We extend Haberland's formula to arbitrary modular forms for finite index subgroups, and we show that it conceals two stronger formulas. One application is an extension of the Eichler-Shimura isomorphism to the entire space of modular forms. We extend the action of Hecke operators to $\Gamma_0(N)$ and $\Gamma_1(N)$, and we prove algebraically that the pairing on period polynomials appearing in Haberland's formula is Hecke equivariant. As a consequence of this proof, we derive two indefinite theta series identities which can be seen as analogues of Jacobi's formula for the theta series associated with the sum of four squares. We give two ways of determining the extra relations satisfied by the even and odd parts of period polynomials associated with cusp forms, which are independent of the period relations.

1. INTRODUCTION

Let Γ be a finite index subgroup of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, and let $S_k(\Gamma)$ be the space of cusp forms of even weight $k \geq 2$ for Γ . Let V_w the Γ -module of complex polynomials of degree at most $w = k - 2$. Each form $f \in S_k(\Gamma)$ determines a collection of polynomials $\rho_f : \Gamma \backslash \Gamma_1 \rightarrow V_w$ given by

$$\rho_f(A) = \int_0^{i\infty} f|_k A(t)(t - X)^w dt.$$

The object ρ_f belongs to the induced Γ_1 -module $\mathrm{Ind}_{\Gamma}^{\Gamma_1} V_w$, and we call it the (multiple) period polynomial associated to f . The goal of this paper is to investigate the structure of the space of period polynomials, reflecting the Petersson inner product and the Hecke operators on modular forms. Working inside the subspace of period polynomials $W_w^{\Gamma} \subset \mathrm{Ind}_{\Gamma}^{\Gamma_1} V_w$, we show that the Petersson product, and the action of Hecke operators for certain Γ , can be stated in a simple way in terms of period polynomials. On an abstract level, this is explained by the fact that the parabolic cohomology class associated to f is completely determined by ρ_f , as reviewed in Section 2 where we restate the Eichler-Shimura isomorphism in terms of period polynomials. Our results can be interpreted as translating the cup product and the action of Hecke operators on cohomology, into a pairing and a Hecke action on the space of period polynomials.

For the full modular group, it was shown by Haberland that the Petersson product of two cusp forms can be expressed in terms of a pairing on their period polynomials [Ha83, KZ84]. In Section 3 we show that Haberland's formula can be extended to finite index subgroups of Γ_1 , and that it splits in two simpler formulas, pairing the opposite parity parts (respectively the same parity parts) of the period polynomials of the two forms when k is even (respectively when k is odd). For the full modular group, the stronger formulas were observed by different means in [Po11]. When

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$k = 2$, a generalization of Haberland’s formula was given by Merel [Me09]. Our proof is simplified by the use of Stokes’ theorem on a fundamental domain for $\Gamma(2)$, which clarifies the appearance of the period polynomial pairing in the formula.

The pairing on period polynomials appearing in Haberland’s formula is degenerate, its radical consisting of the “coboundary polynomials” (Section 4). We show that it becomes nondegenerate after extending it to the larger space of period polynomials of all modular forms. We define the larger period polynomial space \widehat{W}_w^Γ in Section 8, following the Γ_1 construction in [Za91], and we generalize Haberland’s formula and its refinement to the (regularized) Petersson inner product of arbitrary modular forms.¹ For $\Gamma = \Gamma_1(N)$ and $k > 2$, we show that the period polynomial maps extend to isomorphisms between the entire space of modular forms $M_k(\Gamma)$ and the plus and minus eigenspaces of \widehat{W}_w^Γ , under an action corresponding to complex conjugation. This can be seen as an extension of the Eichler-Shimura isomorphism to the entire space of modular forms. Surprisingly, when $k = 2$ the two maps are not always isomorphisms: for $\Gamma = \Gamma_0(N)$ with N square free with at least two prime factors, precisely one of the two maps is an isomorphism (Proposition 8.4 and Remark 8.5).

The action of Hecke operators on period polynomials was defined algebraically by Zagier for the full modular group [Za90, Za93, CZ93]. It was extended by Diamantis to operators of index coprime with the level for the congruence subgroups $\Gamma_0(N)$ [Di01]. We show in Section 5 that the same elements as in the full level case, which go back to work of Manin [Ma73], give a Hecke action on period polynomials for $\Gamma_0(N), \Gamma_1(N)$. We then show that the period polynomial pairing appearing in Haberland’s formula is Hecke equivariant. We give two proofs, one relying on the generalization of Haberland’s formula and the other on an algebraic property of the Hecke elements, proved in Section 9. In Section 8.3 we show that the extended pairing on the space of period polynomials of all modular forms is also Hecke equivariant, for $\Gamma = \Gamma_0(N)$. As an application, we show that the trace of Hecke operators on the space of period polynomials equals the trace on $S_k(\Gamma)$ plus the trace on $M_k(\Gamma)$ (Remark 8.10).

We give two applications of the stronger form of Haberland’s formula for cusp forms: in Section 6 we prove a decomposition of cusp forms in terms of Poincaré series generators; while in Section 7 we obtain the extra relations satisfied by the even and odd period polynomials of cusp forms, obtained by Kohnen and Zagier in the full level case [KZ84]. For $\Gamma_0(N)$, we characterize those N for which the extra relations involve only the even parts of period polynomials just like in the full level case. For $\Gamma_0(N)$, the extra relations are explicit once the period polynomials of the generators with rational periods are computed, as partially done in [An92], [FY09]. For small N that is enough to give completely explicit relations, and we illustrate this for $\Gamma_0(2)$.

A second method of obtaining explicit extra relations in higher generality is given in Section 8.4. It generalizes the Γ_1 approach in [KZ84], by using period polynomials of Eisenstein series and Haberland’s formula for arbitrary modular forms.

The main technical part of the paper is contained in Section 9, where we prove various algebraic properties of the elements defining Hecke operators, which may be of independent interest. An unexpected outcome of this algebraic approach is the discovery of the following “indefinite theta series” identities, which are proved in Section 9.2:

$$(1.1) \quad \sum'_{\substack{a,b,c,d \geq 0 \\ a+b > |d-c|, \ c+d > |a-b|}} q^{ad+bc} + 4 \sum'_{d \geq a > 0} a q^{ad} = 3\tilde{E}_2(q) + \sum_{n > 0} q^{n^2},$$

¹Haberland’s formula has also been generalized to weakly holomorphic modular forms of full level in [BGKO].

$$(1.2) \quad \sum'_{\substack{x \geq |y|, z \geq |t| \\ x > |t|, z > |y|}} q^{x^2+z^2-y^2-t^2} + 2 \sum_{x > |y|} (x - |y|) q^{x^2-y^2} = \tilde{E}_2(q) - 2\tilde{E}_2(q^2) + 4\tilde{E}_2(q^4) + \sum_{n>0} q^{n^2},$$

where $\tilde{E}_2(q) = \sum_{n>0} \sigma_1(n)q^n$ is the weight 2 Eisenstein series without the constant term, and the notation \sum' indicates that the terms for which there is equality in the range are summed with weight $1/2$. For example, when p is an odd prime, we obtain that $p - 3$ is the number of integer solutions of

$$x^2 + z^2 - y^2 - t^2 = p, \text{ with } x, z > |y|, |t|.$$

This can be seen as an indefinite analogue of Jacobi's result on the number of ways of representing a positive integer as a sum of four squares. It would be interesting to fit these examples into a theory of theta series for indefinite quadratic forms, of the type developed by Zwegers [Zw02] for forms of signature $(r, 1)$.

We note that period polynomials are dual to modular symbols, in the sense that the coefficients of period polynomials are values of the integration pairing between modular forms and Manin symbols (the duality between cohomology and homology). The results of this paper can therefore be rephrased in terms of modular symbols. For example, for $\Gamma_1(N)$ the action of Hecke operators on Manin symbols has been determined by Merel [Me94], obtaining the same elements for the operators of index coprime with the level.

2. PERIOD POLYNOMIALS AND THE EICHLER-SHIMURA ISOMORPHISM

The theory of period polynomials for $\Gamma_0(N)$ has been treated in [Sk90, An92, Di01]. We review it here in a general setting, and interpret the Eichler-Shimura isomorphism in terms of period polynomials. We use the properties of the pairing on period polynomials introduced in Section 3 to prove injectivity of the Eichler-Shimura map. In this section we fix notations in use throughout the paper.

Let Γ be a finite index subgroup of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$, and denote by $\bar{\Gamma} = \Gamma/(\Gamma \cap \{\pm 1\})$ the projectivisation of Γ . Throughout the paper, the weight $k \geq 2$ is an integer, and we set $w = k - 2$. Let V_w be the module of complex polynomials of degree at most w , with (right) Γ_1 -action by the $|-_w$ operator: $P|_{-w}g(z) = P(gz)j(g, z)^w$ where $j(g, z) = cz + d$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. Since this is the only action on polynomials, we will omit the subscript.

Viewing V_w as a Γ -module, let \tilde{V}_w^Γ be the induced Γ_1 -module $\mathrm{Ind}_{\Gamma}^{\Gamma_1}(V_w)$. Since V_w is also a Γ_1 -module, we can identify \tilde{V}_w^Γ with the space of maps $P : \Gamma \backslash \Gamma_1 \rightarrow V_w$ with Γ_1 action:

$$P|g(A) = P(Ag^{-1})|g.$$

By the Shapiro isomorphism, we have $H_P^1(\Gamma, V_w) \simeq H_P^1(\Gamma_1, \tilde{V}_w^\Gamma)$ (parabolic cohomology groups). For background on Shapiro's lemma and induced modules, see [NSW, p.59].

Letting $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, for any cocycle $\sigma : \Gamma_1 \rightarrow V_w^\Gamma$ we have $\sigma(g)|1 - J = \sigma(J)(1 - g)$ for all $g \in \Gamma_1$ (which follows from $\sigma(Jg) = \sigma(gJ)$). It follows that the cocycle $\tilde{\sigma} = \frac{\sigma + \sigma|J}{2}$ is in the same cohomology class as σ , where $\sigma|J(g) := \sigma(g)|J$. Since the cocycle $\tilde{\sigma}$ takes values in the subspace

$$V_w^\Gamma := \{P \in \tilde{V}_w^\Gamma : P|J = P, \text{ that is } P(A) = (-1)^w P(-A)\}$$

we have $H_P^1(\Gamma_1, \tilde{V}_w^\Gamma) \simeq H_P^1(\Gamma_1, V_w^\Gamma)$ and from now on we will only work inside the space V_w^Γ . Note that when k is even, V_w is both a $\bar{\Gamma}$ and a $\bar{\Gamma}_1$ module, and the space V_w^Γ can be identified with $\mathrm{Ind}_{\bar{\Gamma}}^{\bar{\Gamma}_1}(V_w)$.

Let now $f \in S_k(\Gamma)$, and define a cocycle $\sigma_f : \Gamma_1 \rightarrow V_w^\Gamma$ by:

$$\sigma_f(g)(A) = \int_{g^{-1}i\infty}^{i\infty} f|A(t)(t-X)^w dt,$$

where the stroke operator acting on modular forms of weight k is $f|g = f|_k g$ for $g \in GL_2(\mathbb{R})^+$. The action of the coset A is defined by acting with any coset representative; this is independent of the representative chosen since $f|_k \gamma = f$ for $\gamma \in \Gamma$. We will show at the end of this section that σ_f satisfies the cocycle relation

$$\sigma_f(g_1 g_2) = \sigma_f(g_2) + \sigma_f(g_1)|g_2.$$

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and let $U = TS$, so that $U^3 = J$. Clearly $\sigma_f(\pm T^n)$ vanishes for $n \in \mathbb{Z}$, and it is easy to see (by a change of variables) that it is a coboundary for other parabolic elements of Γ_1 , hence σ_f defines an element $[\sigma_f] \in H_P^1(\Gamma_1, V_w^\Gamma)$. Since T , S and J generate Γ_1 , it follows that this element is completely determined by the value $\rho_f = \sigma_f(S) \in V_w^\Gamma$, which is the multiple period polynomial attached to f in the introduction. Using the fact that $\sigma_f(S^2) = \sigma_f(U^3) = \sigma_f(US) = 0$ and the cocycle relation, it follows that ρ_f satisfies the period polynomial relations:

$$\rho_f|(1+S) = 0, \quad \rho_f|(1+U+U^2) = 0.$$

We also have $\rho_f(-A) = (-1)^w \rho_f(A)$, so $\rho_f|J = \rho_f$. Therefore the image of the map $f \rightarrow \rho_f$ is contained in the subspace²

$$W_w^\Gamma = \{P \in V_w^\Gamma : P|(1+S) = 0, \quad P|(1+U+U^2) = 0, \quad P|J = P\}$$

whose elements we call *period polynomials* (each element is in fact a collection of $[\Gamma_1 : \Gamma]$ polynomials belonging to V_w).

In fact, setting $C_w^\Gamma = \{P|(1-S) : P \in V_w^\Gamma, \quad P|T = P\} \subset W_w^\Gamma$, we have an isomorphism

$$(2.1) \quad H_P^1(\Gamma_1, V_w^\Gamma) \simeq W_w^\Gamma / C_w^\Gamma,$$

obtained by choosing representative cocycles σ such that $\sigma(T) = \sigma(J) = 0$, and sending $[\sigma]$ to $\sigma(S) \in W_w^\Gamma$. The space C_w^Γ is the image of coboundaries, and we show in Lemma 4.2 that its dimension equals the dimension of the Eisenstein subspace $\mathcal{E}_k(\Gamma)$ of $M_k(\Gamma)$.

Assume now that Γ is normalized by $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The matrix ϵ acts on $P \in V_w^\Gamma$ by

$$(2.2) \quad P|\epsilon(A) = P(A')|_{-w}\epsilon,$$

where $A' = \epsilon A \epsilon$. This action is compatible with the action of Γ_1 : $P|g|\epsilon = P|\epsilon|g\epsilon$ for all $g \in \Gamma_1$. If $f^* \in S_k(\Gamma)$ denotes the form $f^*(z) = \overline{f(-\bar{z})}$, then

$$(2.3) \quad \overline{\rho_{f^*}} = (-1)^{w+1} \rho_f|\epsilon$$

where $\overline{P}(A)$ is obtained by taking the complex conjugates of the coefficients of $P(A)$.

Under the action of ϵ , the space V_w^Γ breaks into ± 1 -eigenspaces, denoted by $(V_w^\Gamma)^\pm$. For $P \in V_w^\Gamma$ we denote its $+1$ and -1 -components by P^+ and P^- respectively:

$$(2.4) \quad P^\pm = \frac{1}{2}(P \pm P|\epsilon) \in (V_w^\Gamma)^\pm.$$

We call P^+ the *even part* and P^- the *odd part* of P , which is justified by the fact that $P(I)^+$ is an even polynomial, and $P(I)^-$ is an odd polynomial (I is the identity coset). For $P \in W_w^\Gamma$, it is easily

²The condition $P|J = P$ is part of the definition of V_w^Γ , but we include it for clarity.

checked that $P|\epsilon \in W_w^\Gamma$ as well. Therefore $P^+, P^- \in W_w^\Gamma$, and the space W_w^Γ also decomposes into eigenspaces $(W_w^\Gamma)^\pm$.

Making use of the pairing on period polynomials introduced in Section 3, we restate the Eichler-Shimura isomorphism in terms of period polynomials as follows.

Theorem 2.1 (Eichler-Shimura). *The two maps $\rho^\pm : S_k(\Gamma) \rightarrow (W_w^\Gamma)^\pm$, $f \rightarrow \rho_f^\pm$, give rise to isomorphisms, denoted by the same symbols:*

$$(2.5) \quad \rho^\pm : S_k(\Gamma) \longrightarrow (W_w^\Gamma)^\pm / (C_w^\Gamma)^\pm.$$

Proof. By the stronger version of Haberland's formula (Theorem 3.3), the two maps $\rho^\pm : S_k(\Gamma) \rightarrow (W_w^\Gamma)^\pm$ are injective. Moreover, their images³ intersect trivially with C_w^Γ by Lemma 4.1, so the two maps in (2.5) are also injective. Using (2.1) and the Eichler-Shimura isomorphism [Sh71, Ch. 8] we have $\dim W_w^\Gamma = 2 \dim S_k(\Gamma) + \dim C_w^\Gamma$, and we conclude that ρ^\pm in (2.5) are isomorphisms. \square

We now show that σ_f satisfies the cocycle relation, while also giving another construction of associated period polynomial. In analogy with the Γ_1 case, the ‘‘Eichler integral’’ associated with $f \in S_k(\Gamma)$ is a function $\tilde{f} : \Gamma \backslash \Gamma_1 \rightarrow \mathcal{A}$, where \mathcal{A} is the space of holomorphic functions on the upper half plane, given by:

$$(2.6) \quad \tilde{f}(A)(z) = \int_z^{i\infty} f|A(t)(t-z)^w dt,$$

with Γ_1 -action as on period polynomials: $\tilde{f}|g(A) = \tilde{f}(Ag^{-1})|_{-w}g$ for $g \in \Gamma_1$. By a change of variables we see that $\tilde{f}|(1-g) = \sigma_f(g)$, which implies that σ_f satisfies the cocycle relation. Note that this provides another construction for the period polynomial ρ_f attached to f , which we record for further use:

$$(2.7) \quad \rho_f = \tilde{f}|(1-S).$$

Remark 2.2. A similar construction will be used in Section 8 to define period polynomials of arbitrary modular forms, by means of an Eichler integral \tilde{f} of $f \in M_k(\Gamma)$, which has the property that $\tilde{f}|_{-w}(1-T) = 0$ and $\tilde{f}|_{-w}(1-S)$ is the (extended) period polynomial attached to f . As pointed out in [DIT10], the construction of period polynomials of cusp forms using their higher order integrals goes back to Poincaré.

3. GENERALIZATION OF HABERLAND'S FORMULA

In [Ha83], Haberland proved a formula expressing the Petersson product of two cusp forms for the full modular group in terms of a pairing on their period polynomials. In this section we extend Haberland's formula to a finite index subgroup Γ of Γ_1 , and we prove a stronger version for subgroups normalized by ϵ .

For $f, g \in S_k(\Gamma)$, define the Petersson scalar product:

$$(f, g) = \frac{1}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2}.$$

On $V_w \times V_w$ we have a natural pairing $\langle \sum a_n x^n, \sum b_n x^n \rangle = \sum (-1)^{w-n} \binom{w}{n}^{-1} a_n b_{w-n}$, satisfying $\langle P, Q \rangle = (-1)^w \langle Q, P \rangle$. We will mostly use the equivalent formulation

$$(3.1) \quad \langle (aX + b)^w, (cX + d)^w \rangle = (ad - bc)^w.$$

³The images of ρ^\pm are determined explicitly in Proposition 7.1 and in Section 8.4.

An easy consequence of (3.1) is that $\langle P|g, Q \rangle = \langle P, Q|g^\vee \rangle$ for $g \in \mathrm{GL}_2(\mathbb{R})$, where $g^\vee = g^{-1} \det g$; in particular the pairing is $\mathrm{SL}_2(\mathbb{R})$ -invariant.

We define a similar pairing on $V_w^\Gamma \times V_w^\Gamma$:

$$(3.2) \quad \langle\langle P, Q \rangle\rangle = \frac{1}{[\bar{\Gamma}_1 : \bar{\Gamma}]} \sum_{A \in \bar{\Gamma} \backslash \bar{\Gamma}_1} \langle P(A), Q(A) \rangle \quad \text{for } P, Q \in V_w^\Gamma.$$

Remark 3.1. For odd k there is a sign ambiguity in defining $P(A)$, $Q(A)$ for $P, Q \in V_w^\Gamma$ and $A \in \bar{\Gamma} \backslash \bar{\Gamma}_1$, but the pairing is well-defined since $P(-A) = (-1)^w P(A)$, $Q(-A) = (-1)^w Q(A)$. For the same reason, one can replace the range by $A \in \Gamma \backslash \Gamma_1$ and the normalizing factor by $\frac{1}{[\Gamma_1 : \Gamma]}$ without changing the pairing. A similar observation applies below, when $f|A$ always appears paired with $\bar{g}|A$, for $f, g \in S_k(\Gamma)$.

This pairing is Γ_1 -invariant: $\langle\langle P|g, Q|g \rangle\rangle = \langle\langle P, Q \rangle\rangle$, for all $P, Q \in V_w^\Gamma$ and $g \in \Gamma_1$. It is normalized such that if $f, g \in S_k(\Gamma)$ and $\Gamma' \subset \Gamma$ then $\langle\langle \rho_f, \rho_g \rangle\rangle_\Gamma = \langle\langle \rho_f, \rho_g \rangle\rangle_{\Gamma'}$. Define also the modified pairing on $V_w^\Gamma \times V_w^\Gamma$:

$$(3.3) \quad \{P, Q\} = \langle\langle P|T - T^{-1}, Q \rangle\rangle,$$

which satisfies $\{P, Q\} = (-1)^{w+1} \{Q, P\}$.

Part a) of the following theorem generalizes Haberland's formula [Ha83, KZ84]. Part b) follows easily from the proof of part a), although to our knowledge it has not appeared previously in the literature (except for Γ_1 in [Po11], but the proof there is more complicated).

Theorem 3.2. a) For $f, g \in S_k(\Gamma)$, we have

$$6C_k \cdot (f, g) = \{\rho_f, \overline{\rho_g}\},$$

where complex conjugation acts coefficientwise on polynomials and $C_k = -(2i)^{k-1}$.

b) For $f, g \in S_k(\Gamma)$, we have $\{\rho_f, \rho_g\} = 0$.

Proof. a) The proof is based on Stokes' theorem, as in [KZ84], except that we apply it to a fundamental domain for $\Gamma(2)$, namely the quadrilateral region \mathcal{D} with vertices $i\infty, -1, 0, 1$ and with sides the geodesics connecting the points in that order. The region \mathcal{D} consists of six copies of the fundamental domain for Γ_1 , which explains the constant 6 appearing in the formula. Therefore we have:

$$(3.4) \quad \begin{aligned} 6C_k[\bar{\Gamma}_1 : \bar{\Gamma}] \cdot (f, g) &= 6 \int_{\Gamma \backslash \mathcal{H}} f(z) \overline{g(z)} (z - \bar{z})^w dz d\bar{z} \\ &= \sum_{A \in \bar{\Gamma} \backslash \bar{\Gamma}_1} \int_{\mathcal{D}} f|A(z) \overline{g|A(z)} (z - \bar{z})^w dz d\bar{z} \\ &= \sum_{A \in \bar{\Gamma} \backslash \bar{\Gamma}_1} \int_{\partial \mathcal{D}} F_A(z) \overline{g|A(z)} d\bar{z} \end{aligned}$$

where $F_A(z) = \int_{i\infty}^z f|_k A(t) (t - \bar{z})^w dt$, so that $\frac{\partial F_A}{\partial z} = f|A(z) (z - \bar{z})^w$, and the last line follows from Stokes' theorem. For $B \in \mathrm{SL}_2(\mathbb{Z})$ a change of variables shows that

$$(3.5) \quad j(B, \bar{z})^w F_A(Bz) = F_{AB}(z) - \int_{i\infty}^{B^{-1}i\infty} f|AB(t) (t - \bar{z})^w dt.$$

We denote by \int_a^b the integral over the geodesic arc from the cusp a to the cusp b . A change of variables $z = T^2\tau$ and (3.5) yields:

$$\int_1^{i\infty} F_A(z) \overline{g|A}(z) d\bar{z} = \int_{-1}^{i\infty} F_{AT^2}(\tau) \overline{g|AT^2}(\tau) d\bar{\tau},$$

and it follows that the sum of integrals over the vertical sides of \mathcal{D} vanishes.

A change of variables $z = S\tau$ and (3.5) yields:

$$\begin{aligned} \int_{-1}^0 F_A(z) \overline{g|A}(z) d\bar{z} &= \int_1^{i\infty} F_{AS}(\tau) \overline{g|AS}(\tau) d\bar{\tau} + \int_1^{i\infty} \int_0^{i\infty} f|AS(t)(t-\bar{\tau})^w \overline{g|AS}(\tau) dt d\bar{\tau}. \\ \int_0^1 F_A(z) \overline{g|A}(z) d\bar{z} &= \int_{i\infty}^{-1} F_{AS}(\tau) \overline{g|AS}(\tau) d\bar{\tau} - \int_{-1}^{i\infty} \int_0^{i\infty} f|AS(t)(t-\bar{\tau})^w \overline{g|AS}(\tau) dt d\bar{\tau}. \end{aligned}$$

When adding the last two equations and summing over $A \in \bar{\Gamma} \backslash \bar{\Gamma}_1$, the single integrals cancel as before and (3.4) becomes

$$6C_k[\bar{\Gamma}_1 : \bar{\Gamma}] \cdot (f, g) = \sum_{A \in \bar{\Gamma} \backslash \bar{\Gamma}_1} \int_1^{i\infty} \int_0^{i\infty} - \int_{-1}^{i\infty} \int_0^{i\infty} f|A(t)(t-\bar{\tau})^w \overline{g|A}(\tau) dt d\bar{\tau}.$$

To write the double integrals in terms of the period polynomial pairing, we use (3.1). After a change of variables $\tau = Tz$ the first double integral becomes

$$\int_0^{i\infty} \int_0^{i\infty} f|A(t) \langle (t-X)^w, (\overline{Tz}-X)^w \rangle \overline{g|AT}(z) dt d\bar{z} = \langle \rho_f(A), \overline{\rho_g}(AT) | T^{-1} \rangle.$$

The second integral yields the same result, with T replaced by T^{-1} , and the conclusion follows from the fact that the pairing $\langle \cdot, \cdot \rangle$ is Γ_1 invariant.

b) Going backwards in the proof of part a) we have:

$$\{\rho_f, \rho_g\} = \frac{1}{[\bar{\Gamma}_1 : \bar{\Gamma}]} \sum_{A \in \bar{\Gamma} \backslash \bar{\Gamma}_1} \int_{\partial \mathcal{D}} H_A(z) g|A(z) dz$$

where $H_A(z) = \int_{i\infty}^z f|_k A(t)(t-z)^w dt = -\tilde{f}(A)(z)$. Since the integrand is now holomorphic and vanishes exponentially at the cusps, it follows that each integral above vanishes. \square

Now let Γ be a congruence subgroup normalized by ϵ . The pairing $\{\cdot, \cdot\}$ satisfies

$$(3.6) \quad \{P|\epsilon, Q|\epsilon\} = (-1)^{w+1} \{P, Q\},$$

hence $\{P, Q\} = 0$ if k is even and $P, Q \in V_w^\Gamma$ have the same parity, or if k is odd and P, Q have opposite parity. We have the following stronger version of Haberland's theorem, generalizing the result for the full modular group from [Pol1].

Theorem 3.3. *Let Γ be a subgroup of finite index in Γ_1 , normalized by ϵ . For $f, g \in S_k(\Gamma)$:*

$$3C_k \cdot (f, g) = \{\rho_f^{\kappa_1}, \overline{\rho_g^{\kappa_2}}\}$$

for any $\kappa_1, \kappa_2 \in \{+, -\}$ with $\kappa_1 \neq \kappa_2$ if k even and $\kappa_1 = \kappa_2$ if k odd.

Proof. We assume k even, the case k odd being entirely similar. In view of Theorem 3.2 a) and (3.6), it is enough to show that $\{\rho_f^+, \overline{\rho_g^+}\} = \{\rho_f^-, \overline{\rho_g^+}\}$. By (2.3), we have $\overline{\rho_g^+} = (-1)^{w+1} \rho_{g^*}^+$, $\overline{\rho_g^-} = (-1)^w \rho_{g^*}^-$, and the previous equality reduces to $\{\rho_f, \rho_{g^*}\} = 0$, which is Theorem 3.2 b). \square

4. COBOUNDARY POLYNOMIALS

In this section we show that the space of coboundary polynomials C_w^Γ is the radical of the bilinear form $\{\cdot, \cdot\}$ on W_w^Γ . The dimension of C_w^Γ equals the dimension of the Eisenstein subspace $\mathcal{E}_k(\Gamma) \subset M_k(\Gamma)$. For $\Gamma = \Gamma_0(N)$, we characterize those N for which $(C_w^\Gamma)^-$ is trivial, namely those N for which the map ρ^- is an isomorphism, as in the full level case.

Lemma 4.1. *Let Γ be a finite index subgroup of Γ_1 . The period polynomials W_w^Γ are orthogonal to the boundary polynomials $C_w^\Gamma \subset W_w^\Gamma$ with respect to the pairing $\{\cdot, \cdot\}$.*

Proof. Let $P|(1-S) \in C_w^\Gamma$ with $P|1-T = 0$, and let $Q \in W_w^\Gamma$. Then $\langle P|(1-S)(T-T^{-1}), Q \rangle = 0$ follows from the following relation in $\mathbb{Z}[\bar{\Gamma}_1]$

$$(4.1) \quad (1-S)(T-T^{-1}) = (T-1)(2+T^{-1}+ST-S) + (1+TS+ST^{-1})(1-S)$$

using the Γ_1 invariance of the pairing $\langle \cdot, \cdot \rangle$ (recall $U = TS$). \square

Let $e_\infty(\Gamma)$, $e_\infty^{\text{reg}}(\Gamma)$ denote the number of inequivalent cusps, respectively regular cusps [DS05, Ch. 3]. The next lemma shows that $\dim C_w^\Gamma = \dim \mathcal{E}_k(\Gamma)$.

Lemma 4.2. *Let Γ be any finite index subgroup of Γ_1 . The dimension of C_w^Γ equals: $e_\infty(\Gamma)$ if $k > 2$ is even; $e_\infty(\Gamma)-1$ if $k = 2$; $e_\infty^{\text{reg}}(\Gamma)$ if $k > 2$ is odd.*

Proof. Recall that $C_w^\Gamma = \{P|(1-S) : P \in V_w^\Gamma \cap \ker(1-T)\}$. Let $P \in V_w^\Gamma \cap \ker(1-T)$. Then $P|T^n = P$ for every $n \in \mathbb{Z}$, that is $P(AT^{-n})|T^n = P(A)$ for $A \in \Gamma \backslash \Gamma_1$. Since $\Gamma \backslash \Gamma_1$ is finite, there is n such that $AT^{-n} = A$, and $P(A)(X+n) = P(A)(X)$. Since the only periodic polynomials are the constants, it follows that $P(A)(X) = c_A \in \mathbb{C}$, with $c_{AT} = c_A$. Since $P|J = P$ we also have $c_{AJ} = (-1)^w c_A$. Hence we have:

$$(4.2) \quad C_w^\Gamma = \{(c_A - c_{AS^{-1}}X^w)_A \in V_w^\Gamma : c_A \in \mathbb{C}, c_{AT} = c_A, c_{AJ} = (-1)^w c_A\}.$$

If $k > 2$ is even it follows that $\dim C_w^\Gamma = |\Gamma \backslash \Gamma_1 / \Gamma_{1\infty}|$, with $\Gamma_{1\infty} = \{\pm T^n : n \in \mathbb{Z}\}$ the stabilizer of ∞ . Since the map

$$\Gamma \backslash \Gamma_1 / \Gamma_{1\infty} \rightarrow \Gamma \backslash \mathbb{P}^1(\mathbb{Q}), \quad [\gamma] \rightarrow [\gamma\infty]$$

is a bijection and $|\Gamma \backslash \mathbb{P}^1(\mathbb{Q})| = e_\infty(\Gamma)$, the claim follows.

If $k = 2$, we identify $\mathbb{C}^{e_\infty(\Gamma)}$ with the vector space $\{(c_A)_{A \in \Gamma \backslash \Gamma_1} : c_{AT} = c_A = c_{AJ} \in \mathbb{C}\}$ and define the (surjective) map $\mathbb{C}^{e_\infty(\Gamma)} \rightarrow C_w^\Gamma$ by $(c_A)_A \rightarrow (c_A - c_{AS})_A$. Its kernel consists of those vectors $(c_A)_A$ with $c_{AT} = c_{AS} = c_{AJ} = c_A$. Since S, T, J generate Γ_1 , it follows that $c_A = c$ for all $A \in \Gamma \backslash \Gamma_1$, so the kernel is isomorphic to \mathbb{C} . Therefore $\dim C_w^\Gamma = e_\infty(\Gamma) - 1$.

If $k > 2$ is odd (so $-1 \notin \Gamma$), we have $c_{AJ} = -c_A$. Therefore $\dim C_w^\Gamma$ equals the number of classes $[A] \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, such that the two associated classes $[A]^+, [AJ]^+ \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}^+$ are distinct, where $\Gamma_{1\infty}^+ = \{T^n : n \in \mathbb{Z}\}$ (when $[A]^+ = [AJ]^+$ then clearly $c_A = c_{AJ} = 0$). But $[A]^+ \neq [AJ]^+$ precisely when $[A]$ corresponds to a regular cusp of Γ since $[A]^+ = [AJ]^+$ means that $A^{-1}\gamma A = -T^n$ for some $\gamma \in \Gamma$, $n > 1$, so the cusp $A\infty$ is irregular. We conclude that $\dim C_w^\Gamma = e_\infty^{\text{reg}}(\Gamma)$. \square

Definition 4.3. Following the proof of Lemma 4.2, we call *cusp* a double coset $\mathcal{C} \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, which corresponds to the Γ -equivalence class of the usual cusp $A\infty$ for any representative $A \in \mathcal{C}$. We call *regular* those cusps $\mathcal{C} = [A]$ such that the double cosets $[A]^+, [AJ]^+ \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}^+$ are distinct. The terminology agrees with the usual one for the cusp $A\infty$ by the last paragraph in the proof of the lemma.

For $\Gamma_0(N)$ it turns out that $(C_w^\Gamma)^-$ is often trivial, in which case ρ^- is an isomorphism just like for Γ_1 . The following proposition was discovered using SAGE [SG].

Proposition 4.4. *Let $\Gamma = \Gamma_0(N)$. Then $(C_w^\Gamma)^- = \{0\}$ if and only if $N = 2^e N'$ with N' odd square free and $0 \leq e \leq 3$.*

Proof. From the proof of Lemma 4.2 we identify $(C_w^\Gamma)^-$ with the space $(\mathbb{C}^{e_\infty(\Gamma)})^-$ of vectors $(c_A)_{A \in \bar{\Gamma} \setminus \bar{\Gamma}_1}$ with $c_A = c_{AT}$ and $c_A = -c_{A'}$ (including for $k = 2$).

Assume N does not satisfy the conditions, so there exists $t \geq 3$ with $t^2 | N$. We claim that $[A] \neq [A']$ for $A = \begin{pmatrix} x & y \\ t & z \end{pmatrix} \in \Gamma_1$, so $(C_w^\Gamma)^- \neq \{0\}$. Assuming by contradiction that $\gamma AT^s = A'$ for $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, it follows that $cx + dt = -t, c(y + sx) + d(z + st) = z$. The first equation implies that $t|d + 1$ while the second that $t|d - 1$, a contradiction with $t \geq 3$.

Assuming N satisfies the conditions, let $(c_A)_{A \in \bar{\Gamma} \setminus \bar{\Gamma}_1}$ with $c_A = c_{AT}$ and $c_A = -c_{A'}$. Identifying the coset space $\Gamma_0(N) \setminus \Gamma_1$ with $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, it follows that $c_{(a:b)} = c_{(a:a+b)}$, $c_{(a:b)} = -c_{(-a:b)}$. The second relation implies $c_{(0:1)} = 0$. Let $N = dd'$ and $k \in \mathbb{Z}$, $(k, d) = 1$. We will show that $c_{(d:k)} = 0$. We have:

$$c_{(d:k)} = c_{(d:k+ad)} = -c_{(-d:k)} = -c_{((bd'-1)d:k)}$$

and it is enough to find $a, b \in \mathbb{Z}$ with $(bd' - 1, N) = 1$ and $k \equiv (bd' - 1)(k + ad) \pmod{N}$. The latter equation can be written $k(bd' - 2) \equiv ad \pmod{dd'}$ and the hypothesis on N ensures that $(d, d') | 2$, so that we can find b, u such that $bd' - 2 = du$. Taking $a \equiv ku \pmod{d'}$, it follows that $(d : k + ad) = ((bd' - 1)d : k)$, which implies that $c_{(d:k)} = 0$. It follows that $c_A = 0$ for all $A \in \bar{\Gamma} \setminus \bar{\Gamma}_1$, finishing the proof. \square

5. HECKE OPERATORS

The action of Hecke operators on period polynomials for the full modular group has been defined algebraically by Choie and Zagier [CZ93] (see also [Za93]). It has been generalized to $\Gamma_0(N)$ for N square free by Antoniadis [An92], but the action defined there is quite complicated. In [Di01], Diamantis showed that an action of Hecke operators of index coprime with the level on period polynomials for $\Gamma_0(N)$ can be defined via the same elements used by Choie and Zagier in the Γ_1 case. Here we extend this construction to all Hecke operators for $\Gamma_0(N)$ and $\Gamma_1(N)$, following the Eichler integral method described in [Za93]. We then prove that the pairing $\{\cdot, \cdot\}$ is Hecke equivariant for Hecke operators of index coprime with the level. For Γ_1 , the Hecke equivariance is mentioned without proof in [GKZ, p. 96].

Let M_n be the set of integer matrices of determinant n , modulo $\{\pm 1\}$, and $R_n = \mathbb{Q}[M_n]$. Thus $\bar{\Gamma}_1$ acts on R_n by left and right multiplication. Let $M_n^\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : n = ad, 0 \leq b < d \right\}$ be the usual system of representatives for $\bar{\Gamma}_1 \setminus M_n$, and $T_n^\infty = \sum_{M \in M_n^\infty} M \in R_n$. Following [CZ93], let $\tilde{T}_n, Y_n \in R_n$ be such that

$$(5.1) \quad T_n^\infty(1 - S) = (1 - S)\tilde{T}_n + (1 - T)Y_n.$$

We will show that for all n , the elements \tilde{T}_n define a Hecke action on period polynomials for $\Gamma_0(N)$ or $\Gamma_1(N)$. Note that the elements \tilde{T}_n are universal, not depending on the weight or level.

Let Γ be $\Gamma_0(N)$ or $\Gamma_1(N)$, and let Δ_n be the union of double cosets defining the Hecke operator T_n on $f \in S_k(\Gamma)$, namely Δ_n is the set of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ with $\det \gamma = n$ and $N|c$, $(a, N) = 1$ (for $\Gamma_0(N)$), or $N|c, N|(a - 1)$ (for $\Gamma_1(N)$). Let M_n^Γ be a fixed system of representatives for $\Gamma \setminus \Delta_n$, so that $f|T_n = n^{w+1} \sum_{M \in M_n^\Gamma} f|_k M$.

We define an action of M_n on V_w^Γ which depends on Δ_n , and which is based on the following property of the pair (Γ, Δ_n)

(H) The map $\Gamma \backslash \Delta_n \rightarrow \Gamma_1 \backslash \Gamma_1 \Delta_n$, $\Gamma \delta \rightarrow \Gamma_1 \delta$ is bijective.

More generally, one can define an action of Hecke operators associated with double cosets Δ for more general groups Γ , as long as the pair (Γ, Δ) satisfies this property. See [Sh71, Prop. 3.36] for a more general setting when this property holds.

For $A \in \Gamma_1$ and $MA^{-1} \in \Gamma_1 \Delta_n$, by (H) there is a unique decomposition $MA^{-1} = A_M^{-1} M_A$, with $M_A \in M_n^\Gamma$, $A_M \in \Gamma_1$. Since $\Gamma \Delta_n = \Delta_n \Gamma$, the coset ΓA_M depends only on ΓA , so we can define for $P \in V_w^\Gamma$

$$P|M(A) = \begin{cases} P(A_M)|_{-w} M & \text{if } MA^{-1} \in \Gamma_1 \Delta_n \\ 0 & \text{otherwise.} \end{cases}$$

Note that when $(n, N) = 1$, the second case does not occur as $\Gamma_1 \Delta_n$ consists of all integral matrices of determinant n . The action depends on Δ_n (but not on the system of representatives M_n^Γ chosen), so it should be properly denoted $P|_{\Delta_n} M$. For simplicity we omit the dependence from the notation, until it becomes necessary later on.

We use the same notation as for the action of Γ_1 since for $M \in \Gamma_1$ we have $M_A = 1$ for all A and the two actions coincide. Moreover the two actions are compatible: for $h \in \Gamma_1, M \in M_n$ a formal computation shows that

$$(5.2) \quad P|M|h = P|Mh, \quad P|h|M = P|hM, \quad P|M|\epsilon = P|\epsilon|M\epsilon.$$

This action extends linearly to an action of R_n on V_w^Γ . In the same way we define an action of M_n and R_n on the Eichler integrals \tilde{f} in (2.6).

Proposition 5.1. *Let Γ be $\Gamma_0(N)$ or $\Gamma_1(N)$. Then any element \tilde{T}_n as in (5.1) gives an action of Hecke operators on ρ_f for $f \in S_k(\Gamma)$, namely $\rho_f|_{T_n} = \rho_f|\tilde{T}_n$.*

Proof. Using the fact that $\rho_f = \tilde{f}|(1 - S)$ and $\tilde{f}|(1 - T) = \sigma_f(T) = 0$, we have as in [Za93]

$$\rho_f|\tilde{T}_n = \tilde{f}|(1 - S)\tilde{T}_n = \tilde{f}|T_n^\infty(1 - S) = \widetilde{\tilde{f}|T_n}(1 - S) = \rho_f|_{T_n},$$

once we show that $\tilde{f}|T_n^\infty = \widetilde{\tilde{f}|T_n}$.

For $M \in M_n^\infty$ and $A \in \Gamma_1$, let $MA^{-1} = A_M^{-1} M_A$ with $A_M \in \Gamma_1$, $M_A \in M_n^\Gamma$, so that

$$(5.3) \quad \begin{aligned} \tilde{f}|T_n^\infty(A) &= \sum_{M \in M_n^\infty \cap \Gamma_1 \Delta_n A} \int_{Mz}^{i\infty} f|A_M(t)(t - Mz)^w j(M, z)^w dt \\ &= n^{w+1} \sum_{M \in M_n^\infty \cap \Gamma_1 \Delta_n A} \int_z^{i\infty} f|M_A A(u)(u - z)^w du \end{aligned}$$

where we made a change of variables $t = Mu$. For fixed A , the map $M \rightarrow M_A$ is a bijection from $M_n^\infty \cap \Gamma_1 \Delta_n A$ onto M_n^Γ : surjectivity is obvious while injectivity follows from (H). Hence the last expression equals $\widetilde{\tilde{f}|T_n}(A)$ finishing the proof. \square

Corollary 5.2. *With the hypothesis of Proposition 5.1, we have $\rho_f^\pm|_{T_n} = \rho_f^\pm|\tilde{T}_n$.*

Proof. Since the action of M_n on V_w^Γ is compatible with the action of ϵ , it is enough to show that $\epsilon\tilde{T}_n\epsilon$ also satisfies (5.1) (for a different Y_n). This follows from conjugating (5.1) by ϵ , and using that $T_n^\infty - \epsilon T_n^\infty \epsilon \in (1 - T)R_n$. \square

Elements \tilde{T}_n satisfying condition (5.1) go back to work of Manin [Ma73], see for example [CZ93, Za90, Me94]. The element \tilde{T}_n is unique, up to addition of any element in the right Γ_1 -module

$$(5.4) \quad \mathcal{I} = (1 + S)R_n + (1 + U + U^2)R_n.$$

Next we find the adjoint of the action of Hecke operators for the pairing on W_w^Γ defined in (3.3). For $g \in M_n$ we denote by $g^\vee = g^{-1} \det g$ the adjoint of g , and we apply this notation to all elements of R_n by linearity. Recall that $\langle P|g, Q \rangle = \langle P, Q|g^\vee \rangle$ for $P, Q \in V_w$, $g \in GL_2(\mathbb{C})$.

To generalize this property to the action of M_n on V_w^Γ , let $\Delta_n^\vee = \{g^\vee : g \in \Delta_n\}$ be the double coset giving the adjoint Hecke operator T_n^* on modular forms. Assuming $(n, N) = 1$, the pair (Γ, Δ_n^\vee) satisfies Property (H), and we have $\Gamma_1 \Delta_n = \Gamma_1 \Delta_n^\vee = M_n$. Fixing a system of representatives $(M_n^\Gamma)^\vee$ for $\Gamma \backslash \Delta_n^\vee$, we define as above an action $P|_{\text{adj}} M := P|_{\Delta_n^\vee} M$ of $M \in M_n$ on $P \in V_w^\Gamma$. Note that for $\Gamma = \Gamma_0(N)$ with $(n, N) = 1$, we have $\Delta_n = \Delta_n^\vee$ and this is the same as the previous action.

Lemma 5.3. *Let Γ be $\Gamma_0(N)$ or $\Gamma_1(N)$ and assume $(n, N) = 1$. For $P, Q \in V_w^\Gamma$, $M \in M_n$ we have*

$$(5.5) \quad \langle\langle P|M, Q \rangle\rangle = \langle\langle P, Q|_{\text{adj}} M^\vee \rangle\rangle.$$

Proof. For $A \in \Gamma_1$ and $M \in \Gamma_1 \Delta_n A = M_n$, let $MA^{-1} = A_M^{-1} M_A$ with $A_M \in \Gamma_1$, $M_A \in M_n^\Gamma$. We have (see Remark 3.1)

$$[\Gamma_1 : \Gamma] \langle\langle P|M, Q \rangle\rangle = \sum_{A \in \Gamma \backslash \Gamma_1} \langle P(A_M), Q(A)|M^\vee \rangle.$$

Taking adjoint we have $M^\vee A_M^{-1} = A^{-1} M_A^\vee = A^{-1} \gamma M'_A$ with $\gamma \in \Gamma$, $M'_A \in (M_n^\Gamma)^\vee$. Since $(n, N) = 1$, the map $A \rightarrow A_M$ is a bijection of $\Gamma \backslash \Gamma_1$, and summing over A_M instead of A finishes the proof. \square

Next we give two proofs of the Hecke equivariance of the period polynomial pairing, one of them requiring the following lemma.

Lemma 5.4. *The space C_w^Γ is preserved by the Hecke operators \tilde{T}_n , whenever an action of M_n on V_w^Γ satisfying (5.2) can be defined.*

Proof. Let $P|(1 - S) \in C_w^\Gamma$ with $P|1 - T = 0$. By (5.1), $P|(1 - S)|\tilde{T}_n = P|T_n^\infty(1 - S)$, and the latter is an element of C_w^Γ , since $T_n^\infty(1 - T) \in (1 - T)R_n$, so $P|T_n^\infty(1 - T) = 0$. \square

Theorem 5.5. *Let Γ be $\Gamma_0(N)$ or $\Gamma_1(N)$ and assume $(n, N) = 1$. For $P, Q \in W_w^\Gamma$ and any \tilde{T}_n as in (5.1) we have:*

$$\{P|\tilde{T}_n, Q\} = \{P, Q|_{\text{adj}} \tilde{T}_n\}.$$

Proof. We give two proofs. For the first, we assume for simplicity that $\Gamma = \Gamma_0(N)$, so that the action $|_{\text{adj}}$ is the same as $|$. From Theorem 3.3 and the Hecke equivariance of the Petersson inner product, it follows that the claim is true for $P = \rho_f^\pm$ and $Q = \rho_g^\mp$, for any $f, g \in S_k(\Gamma)$. By (2.5), any $P \in W_w^\Gamma$ can be written $P = \rho_f^+ + \rho_g^- + Q$ with $Q \in C_w^\Gamma$. Since \tilde{T}_n preserves $(W_w^\Gamma)^\pm$ and C_w^Γ , the claim follows taking into account Lemmas 4.1, 5.4 and (3.6).

The second proof is purely algebraic. Via (5.5) the equality to prove is equivalent to

$$\langle\langle P|[\tilde{T}_n(T - T^{-1}) + (T^{-1} - T)\tilde{T}_n^\vee], Q \rangle\rangle = 0.$$

Since $P, Q \in W_w^\Gamma$, we are reduced to proving the next theorem. \square

Theorem 5.6. *For any element $\tilde{T}_n \in R_n$ satisfying property (5.1) we have*

$$\tilde{T}_n(T - T^{-1}) + (T^{-1} - T)\tilde{T}_n^\vee \in \mathcal{I} + \mathcal{I}^\vee.$$

The proof is quite involved, and we postpone it to Section 9.

6. RATIONAL DECOMPOSITION OF MODULAR FORMS

For Γ a finite index subgroup of Γ_1 normalized by ϵ , we give an explicit decomposition of $f \in S_k(\Gamma)$ in terms of explicit generators, generalizing the result in the full level case [Po11]. For Γ_1 , these are the generators with rational periods studied in [KZ84], where their periods were first computed. For $\Gamma_0(N)$, the periods of these generators were computed for N square free in [An92], and for arbitrary N in [FY09]. These generators have explicit formulas as Poincaré series when $k > 2$.

To define these generators, for $A \in \Gamma \backslash \Gamma_1$, $0 \leq n \leq w$, define the periods $r_{A,n}(f)$ by

$$(6.1) \quad \rho_f(A)(X) = \sum_{n=0}^w (-1)^{w-n} \binom{w}{n} r_{A,n}(f) X^{w-n}$$

and similarly define $r_{A,n}^\pm(f)$ with ρ_f replaced by ρ_f^\pm . Let $R_{A,n} \in S_k(\Gamma)$ be the dual of the linear functional $f \rightarrow \frac{1}{[\Gamma_1 : \Gamma]} r_{A,n}(f)$, with respect to the Petersson product:

$$(f, R_{A,n}) = \frac{r_{A,n}(f)}{[\Gamma_1 : \Gamma]}, \text{ for all } f \in S_k(\Gamma),$$

and similarly define $R_{A,n}^+$, $R_{A,n}^-$. For $\Gamma = \Gamma_0(N)$, the polynomials $\rho^-(R_{A,n}^+)$, $\rho^+(R_{A,n}^-)$ have rational coefficients.

For $\kappa \in \{+, -\}$ and $0 \leq j \leq w$, define the linear combinations of periods:

$$s_{A,j}^\kappa(f) = \sum_{n=0}^j \binom{j}{n} (-1)^{j-n} r_{A,j}^\kappa(f).$$

We then have the following generalization of Theorem 1.1 in [Po11], which gives explicit inverses of the Eichler-Shimura maps (2.5).

Theorem 6.1. *Let Γ be a finite index subgroup of Γ_1 normalized by ϵ , and let $\kappa_1, \kappa_2 \in \{+, -\}$ with $\kappa_1 \neq \kappa_2$ if k even and $\kappa_1 = \kappa_2$ if k odd. For $f \in S_k(\Gamma)$*

$$\frac{-3C_k}{2} \cdot f = \sum_{A \in \Gamma \backslash \Gamma_1} \sum_{n=0}^w \binom{w}{n} s_{AU^{-1},n}^{\kappa_1}(f) R_{A,n}^{\kappa_2}.$$

Proof. Let $P \in (W_w^\Gamma)^{\kappa_1}$, $Q \in (W_w^\Gamma)^{\kappa_2}$. Then

$$\{P, \overline{Q}\} = \langle\langle P|US - SU^2, \overline{Q} \rangle\rangle = \langle\langle P|U^2 - U, \overline{Q} \rangle\rangle = -2 \langle\langle P|U, \overline{Q} \rangle\rangle$$

where the second equality follows since $P|S = -P$, $Q|S = -Q$, while the third from $P|U^2 - U = P| -1 - 2U$, together with (3.6).

If $R(X) = \sum_{n=0}^w (-1)^{w-n} \binom{w}{n} r_n X^{w-n} \in V_w$ then

$$R|U(X) = \sum_{j=0}^w \binom{w}{j} s_j X^j, \text{ with } s_j = \sum_{n=0}^j (-1)^{j-n} \binom{j}{n} r_n.$$

By Theorem 3.3 and the preceding computations it follows that for $f, g \in S_k(\Gamma)$.

$$3C_k(f, g) = \{\rho_f^{\kappa_1}, \overline{\rho_g^{\kappa_2}}\} = -\frac{2}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_{A \in \overline{\Gamma} \setminus \overline{\Gamma}_1} \sum_{j=0}^w \binom{w}{j} s_{AU^{-1}, j}^{\kappa_1}(f) \overline{r_{A, j}^{\kappa_2}(g)}$$

Since $\overline{r_{A, j}^+}(g) = [\overline{\Gamma}_1 : \overline{\Gamma}] \cdot (R_{A, j}^+, g)$, the claim follows. \square

7. EXTRA RELATIONS SATISFIED BY PERIOD POLYNOMIALS OF CUSP FORMS

In this section, we determine the image of the maps $\rho^\pm : S_k(\Gamma) \rightarrow (W_w^\Gamma)^\pm$, namely the extra relations satisfied by ρ_f^\pm for $f \in S_k(\Gamma)$ which are independent of the period relations. To be explicit, the extra relations we obtain require the determination of the periods $r_{B, m}^\mp(R_{A, w}^\pm)$ of the generators defined in the previous section. For Γ_1 , these periods were computed in [KZ84], and for $\Gamma_0(N)$, they were computed in [An92] (for N square free, and not quite in closed form), and in [FY09] (only the principal periods $r_{I, m}^\mp(R_{I, n}^\pm)$). For $\Gamma_0(N)$ with small N , the computations in [FY09] are sufficient to make completely explicit the extra relations, and we illustrate this for the case $\Gamma = \Gamma_0(2)$. The relations are similar to the relation found by Kohnen and Zagier in the full-level case [KZ84] (see also [Po11, Sec. 2]).

We first define bases of $(C_w^\Gamma)^\pm$, using the terminology in Definition 4.3. For each cusp $\mathcal{C} \in \Gamma \setminus \Gamma_1 / \Gamma_{1\infty}$, which is regular if k is odd, define $P_{\mathcal{C}} \in C_w^\Gamma$ as in (4.2) by fixing $A_{\mathcal{C}} \in \mathcal{C}$ a representative, and setting $c_A = (-1)^w c_{A, J} = 1$ if $[A]^+ = [A_{\mathcal{C}}]^+$, and $c_A = 0$ if $[A] \neq [A_{\mathcal{C}}]$. Then $\{P_{\mathcal{C}}\}$ form a basis of C_w^Γ if $k > 2$, and if $k = 2$ there is only one relation $\sum_{\mathcal{C}} P_{\mathcal{C}} = 0$.

Assume that Γ is normalized by ϵ , and denote $A' = \epsilon A \epsilon$. Note that if $\mathcal{C} = [A]$ is a cusp, then $\mathcal{C}' = [A']$ is well-defined. Since $P_{\mathcal{C}}|_{\epsilon} = P_{\mathcal{C}'}$, we have $P_{\mathcal{C}}^+ = P_{\mathcal{C}'}^+$ and $P_{\mathcal{C}}^- = -P_{\mathcal{C}'}^-$. Therefore a basis of $(C_w^\Gamma)^-$ consists of $P_{\mathcal{C}}^-$ for each unordered pair $(\mathcal{C}, \mathcal{C}')$ of cusps with $\mathcal{C} \neq \mathcal{C}'$, and a basis of $(C_w^\Gamma)^+$ consists of $P_{\mathcal{C}}^+$ for each unordered pair $(\mathcal{C}, \mathcal{C}')$ of cusps (when k is odd only pairs of regular cusps are considered; note that \mathcal{C} is regular iff \mathcal{C}' is regular).

We now fix $A \in \mathcal{C}$ for each (regular if k odd) cusp $\mathcal{C} \in \Gamma \setminus \Gamma_1 / \Gamma_{1\infty}$ and we write, by a slight abuse of notation, $R_{\mathcal{C}, n} = R_{A, n}$, $r_{\mathcal{C}, n} = r_{A, n}$ with the notation of the previous section ($R_{\mathcal{C}, n}$ does depend on the choice of representative A , but we fix such a choice). For each unordered pair of cusps $(\mathcal{C}, \mathcal{C}')$ we take $g = R_{\mathcal{C}, w}^+$ in Theorem 3.3, and if $\mathcal{C} \neq \mathcal{C}'$ we take also $g = R_{\mathcal{C}, w}^-$ (again only for regular cusps if k is odd), obtaining the following linear relations satisfied by all $f \in S_k(\Gamma)$ ⁴

$$(7.1) \quad \frac{3C_k}{[\overline{\Gamma}_1 : \overline{\Gamma}]} r_{\mathcal{C}, w}^+(f) = \{\rho^+(f), \overline{\rho^-(R_{\mathcal{C}, w}^+)}\}, \quad \frac{3C_k}{[\overline{\Gamma}_1 : \overline{\Gamma}]} r_{\mathcal{C}, w}^-(f) = \{\rho^-(f), \overline{\rho^+(R_{\mathcal{C}, w}^-)}\}.$$

The linear forms appearing in these relations can be applied to all $P \in (W_w^\Gamma)^\pm$, and putting together the relations involving $\rho^+(f)$ into a map λ_+ , and the relations involving $\rho^-(f)$ (for pairs with $\mathcal{C} \neq \mathcal{C}'$) into a map λ_- , we obtain two linear maps $\lambda_\pm : (W_w^\Gamma)^\pm \rightarrow \mathbb{C}^{d_\pm}$ with $d_\pm = \dim(C_w^\Gamma)^\pm$. If $d_- = 0$ the map λ_- is trivial.

Proposition 7.1. *Assume $k \geq 3$ and let Γ be a finite index subgroup of Γ_1 normalized by ϵ . With λ_\pm defined above, we have exact sequences:*

$$0 \rightarrow S_k(\Gamma) \xrightarrow{\rho^\pm} (W_w^\Gamma)^\pm \xrightarrow{\lambda_\pm} \mathbb{C}^{d_\pm} \rightarrow 0.$$

⁴In this section we occasionally write $\rho(f)$ instead of ρ_f to simplify notation.

Proof. We have $\text{Im}\rho^\pm \subset \ker \lambda_\pm$ by construction. Note that the first relation in (7.1) is not satisfied by P_C^+ , while the second is not satisfied by P_C^- if $C \neq C'$, since the LHS is nonzero, while the RHS vanishes by Lemma 4.1 a). Since $\{P_C^\pm\}$ form a basis of $(C_w^\Gamma)^\pm$, the conclusion follows. \square

Remark 7.2. For $\Gamma = \Gamma_0(N)$, Proposition 4.4 characterizes those N for which the extra relations involve only the even parts of the period polynomials.

For example, if $\Gamma = \Gamma_0(p)$ with p prime, there are only two cusps $[I]$ and $[S]$, and for all $P \in W_w^\Gamma$ we have $P(S) = -P(I)|S$ by the period relations. In particular $r_{S,w}(f) = -r_{I,0}(f)$. Noting also that $P(I)^+, P(S)^+$ are the even parts of $P(I), P(S)$, so that $R_{I,n}^+ = R_{I,n}$ for n even, we have the following simpler version of Proposition 7.1.

Corollary 7.3. *Let $\Gamma = \Gamma_0(p)$ with p prime, and let $k > 2$ even. Then the two extra relations satisfied by all even period polynomials $\rho^+(f)$ for $f \in S_k(\Gamma)$ are*

$$\frac{3C_k}{[\bar{\Gamma}_1 : \bar{\Gamma}]} r_{I,a}(f) = \{\rho^+(f), \overline{\rho^-(R_{I,a})}\}, \quad \text{for } a = 0, w.$$

For small values of N (eg $N = 2, 3, 4, 5$), the polynomials ρ_f are completely determined by the principal parts $\rho_f(I)$, so that the relations above are completely explicit via the computation of $\rho^-(R_{I,a})(I)$ in [FY09]. In the remainder of this section, we discuss in detail the case of period polynomials for $\Gamma = \Gamma_0(2)$, which have been studied in [IK05], [FY09], [KT11]. The case $\Gamma_0(4) \simeq \Gamma(2)$ is particularly interesting, and will be considered in a future work.

We take as coset representatives for $\Gamma \backslash \Gamma_1$ the set $\{I, U, U^2\}$. Denoting by \bar{A} the coset ΓA , we have $\bar{S} = \bar{U}$, $\bar{U}\bar{S} = \bar{I}$, $\bar{U}^2\bar{S} = \bar{U}^2$. For $P \in W_w^\Gamma$, the period relations $P|1 + S = 0$, $P|1 + U + U^2 = 0$ reduce to

$$(7.2) \quad P(U) + P(I)|S = 0, \quad P(U^2)|1 + S = 0, \quad P(U^2) + P(U)|U + P(I)|U^2 = 0.$$

The polynomials $P(U), P(U^2)$ are therefore determined by $P(I)$ which satisfies the relation

$$(7.3) \quad P(I)|(ST - ST^{-1})(1 + S) = 0.$$

Let $U_w \subset V_w$ denote the set of polynomials satisfying (7.3), so that we can identify W_w^Γ with U_w via $P \rightarrow P(I)$. Conjugation by ϵ leaves unchanged each coset $\bar{I}, \bar{U}, \bar{U}^2$, hence P^+, P^- correspond to the even and odd parts of the polynomial $P(I)$ in this identification.

To express the formula in Theorem 3.3 in terms of $\rho_f(I), \rho_g(I)$ alone, let $P \in (W_w^\Gamma)^+, Q \in (W_w^\Gamma)^-$. We have $\langle\langle P|T - T^{-1}, Q \rangle\rangle = -2 \langle\langle P|U, Q \rangle\rangle$ as in the proof of Theorem 6.1, and using (7.2) we obtain

$$\langle\langle P|U, Q \rangle\rangle = \frac{1}{3} \langle P(I)|2T^{-1} - 2I - T, Q(I) \rangle = -\frac{1}{2} \langle P(I)|T - T^{-1}, Q(I) \rangle$$

where $\langle P(I), Q(I) \rangle = \langle P(I)|T + T^{-1}, Q(I) \rangle = 0$ since $P(I), Q(I)$ have opposite parity. We can therefore restate Theorem 3.3 as follows, setting $P_f = \rho_f(I) \in U_w$

$$(7.4) \quad 3C_k \cdot (f, g) = \langle P_f^+|T - T^{-1}, \overline{P_g^-} \rangle.$$

By Proposition 4.4 and the Eichler-Shimura isomorphism (2.5), the map $f \rightarrow P_f^-$ gives an isomorphism $S_k(\Gamma_0(2)) \simeq U_w^-$, while the image of the map $f \rightarrow P_f^+$ is a codimension 2 subspace of U_w^+ .⁵ To simplify notation, let $r_n(f) = r_{I,n}(f)$, and $R_n = R_{I,n}$ for $0 \leq n \leq w$.

⁵A direct proof in this case is contained in [KT11, Theorem 4].

Corollary 7.4. *For f in $S_k(\Gamma_0(2))$, let $s_n(f) = \sum_{\substack{j=0 \\ n-j \text{ odd}}}^n \binom{n}{j} r_j(f)$. The extra relations satisfied by the even periods of f are*

$$r_a(f) = \sum_{\substack{n=0 \\ n \text{ odd}}}^w \binom{w}{n} s_{w-n}(f) \frac{2}{C_k} r_a(R_n) \quad \text{for } a = 0, w.$$

From [FY09], for $0 < w < n$, n odd, we have $r_w(R_n) = -\frac{r_0(R_{\tilde{n}})}{N^n}$ and

$$\frac{2}{C_k} r_0(R_n) = -N^{\tilde{n}} \frac{B_{\tilde{n}+1}}{\tilde{n}+1} + \frac{k}{B_k} \frac{B_{n+1}}{n+1} \frac{B_{\tilde{n}+1}}{\tilde{n}+1} \frac{\alpha_{N,k}(n)}{N} + \frac{\delta_{w,n+1}}{w},$$

where $\tilde{n} = w - n$, $\alpha_{N,k}(n) = \frac{1-N^{-n-1}}{1-N^{-k}}$ (recall $N = 2$), and B_m are the Bernoulli numbers. Note that there is a minus sign missing in the normalization of the generators denoted by $R_{\Gamma,w,n}$ in [FY09, Def. 1.1], and with this correction we have $R_n = -\frac{C_k}{2} R_{\Gamma,w,n}$.

Proof. Since $P_f|T - T^{-1}(X) = -2 \sum_{n=0}^w (-1)^n \binom{w}{n} s_n(f) X^{w-n}$, and $\overline{r_n(R_0)} = (R_n, R_0) = r_0(R_n)$, the claim follows from Corollary 7.3. \square

When f is a newform, the periods $r_n(f)$ are critical values of the L -series associated to f , and they can be readily computed using MAGMA [Mgm]. The relations in Corollary 7.4 have been checked numerically for $k = 8, 10, 14$.

8. PERIOD POLYNOMIALS OF ARBITRARY MODULAR FORMS

In this section we define period polynomials for noncuspidal modular forms, and extend Haberland's formula and the action of Hecke operators to the larger space of period polynomials of all modular forms. An important feature of the larger space \widehat{W}_w^Γ is that the pairing $\{\cdot, \cdot\}$ has a natural nondegenerate extension to it, while on W_w^Γ it is degenerate (its radical is C_w^Γ). If $\Gamma = \Gamma_1(N)$ and $k > 2$, the period polynomial maps ρ^\pm extend naturally to the larger space, and they give isomorphisms between $M_k(\Gamma)$ and $(\widehat{W}_w^\Gamma)^\pm$. Surprisingly, when $k = 2$, $\Gamma = \Gamma_0(N)$ and N is squarefree with at least two prime factors, only one of the two maps is an isomorphism.

For the full modular group, period polynomials of Eisenstein series were defined in [KZ84], using the description of periods as special values of the associated L -function, and the enlarged space of period polynomials was introduced in [Za91]. A different construction using an Eichler integral was given more recently in [BGKO], in the more general context of weakly holomorphic modular forms. We extend both the Eichler integral and the L -function approach to a finite index subgroup Γ of Γ_1 .

For $f \in M_k(\Gamma)$, we define $\widehat{\rho}_f = \widetilde{f}|1 - S$ as in (2.7) (with the same action as on period polynomials), where the Eichler integral $\widetilde{f} : \Gamma \backslash \Gamma_1 \rightarrow \mathcal{A}$ is given by

$$(8.1) \quad \widetilde{f}(A)(z) = \int_z^{i\infty} [f|A(t) - a_0(f|A)](t - z)^w dt$$

with $a_0(f|A)$ the constant term of the Fourier expansion of $f|A$. Note that $a_0(f|A) = a_0(f|AT)$, so $\tilde{f}|1 - T = 0$. Let

$$\widehat{V}_w := \left\{ \sum_{-1 \leq i \leq w+1} a_i X^i : a_i \in \mathbb{C} \right\},$$

and let \widehat{V}_w^Γ be the space of functions $P : \Gamma \backslash \Gamma_1 \rightarrow \widehat{V}_w$ with $P(-A) = (-1)^w P(A)$ for $A \in \Gamma \backslash \Gamma_1$. We define an action of $g \in \Gamma_1$ on $P \in \widehat{V}_w^\Gamma$ by $P|g(A) = P(Ag^{-1})|_{-w}g$ as before. Note that this is no longer well defined in general, as elements of Γ_1 do not preserve the space \widehat{V}_w . However one can still define the subspace

$$\widehat{W}_w^\Gamma = \{P \in \widehat{V}_w^\Gamma : P|1 + S = P|1 + U + U^2 = 0\}.$$

We will show below that $\widehat{\rho}_f \in \widehat{W}_w^\Gamma$. That it satisfies the period relations is immediate: we have $\widehat{\rho}_f|1 + S = 0$, and $\widehat{\rho}_f|1 - T = 0$ from the definition, while

$$\widehat{\rho}_f|1 + U + U^2 = \tilde{f}|(1 - S)(1 + U + U^2) = \tilde{f}|(1 - T^{-1})(1 + U + U^2) = 0.$$

It remains to show that $\widehat{\rho}_f \in \widehat{V}_w^\Gamma$, and we will do this by relating it with the polynomial $\rho_f \in V_w^\Gamma$ defined by (6.1), where

$$(8.2) \quad r_{A,n}(f) = (-1)^{n+1} \frac{\Gamma(n+1)}{(2\pi i)^{n+1}} L(n+1, f|A).$$

The L -function $L(s, f) = \sum_{n=1}^{\infty} a_n(f) n^{-s}$ is given, if $\operatorname{Re}(s) > k$, by the Mellin transform

$$(-1)^s (2\pi i)^{-s} \Gamma(s) L(s, f) = \int_0^{i\infty} [f(t) - a_0(f)] t^{s-1} dt,$$

and it can be extended meromorphically to \mathbb{C} by fixing $z_0 \in \mathcal{H}$, decomposing $\int_0^{i\infty} = \int_0^{z_0} + \int_{z_0}^{i\infty}$, and making a change of variables $t = Su$ in the first integral. We obtain a meromorphic function with at most simple poles at $s = 0$ and $s = k$:

$$\begin{aligned} \frac{(-1)^s \Gamma(s)}{(2\pi i)^s} L(s, f) &= \int_{z_0}^{i\infty} [f(t) - a_0(f)] t^{s-1} dt + (-1)^s \int_{S z_0}^{i\infty} [f|S(t) - a_0(f|S)] t^{k-s-1} dt - \\ &\quad - a_0(f) \frac{z_0^s}{s} - (-1)^s a_0(f|S) \frac{(S z_0)^{k-s}}{k-s}. \end{aligned}$$

Introducing as in [KZ84] the function $H_{z_0} \in V_w^\Gamma$ defined for $A \in \Gamma \backslash \Gamma_1$ by

$$(8.3) \quad H_{z_0}(A) = \int_{z_0}^{i\infty} [f|A(t) - a_0(f|A)] (t - X)^w dt - a_0(f|A) \int_0^{z_0} (t - X)^w dt \in V_w$$

we obtain from (6.1) and the analytic continuation above

$$(8.4) \quad \rho_f(A) = H_{z_0}(A) - H_{S z_0}(AS^{-1})|S,$$

namely $\rho_f = H_{z_0} - H_{S z_0}|S$.

We now determine the relation between $\widehat{\rho}_f$ and ρ_f , which also shows that $\widehat{\rho}_f \in \widehat{W}_w^\Gamma$.

Proposition 8.1. *For $f \in M_k(\Gamma)$, let $\rho_f^0 \in \widehat{V}_w^\Gamma$ be given by $\rho_f^0(A) = (-1)^w \frac{a_0(f|A)}{w+1} X^{w+1}$. We have*

$$\widehat{\rho}_f = \rho_f + \rho_f^0(1 - S),$$

namely $\widehat{\rho}_f(A) = \rho_f(A) + (-1)^w \frac{a_0(f|A)}{w+1} X^{w+1} + \frac{a_0(f|AS^{-1})}{w+1} X^{-1}$.

Proof. Fixing $z_0 \in \mathcal{H}$ and decomposing the integral in (8.1) as $\int_{z_0}^{i\infty} + \int_z^{z_0}$ we have

$$\tilde{f}(A)(z) = H_{z_0}(A)(z) + \int_z^{z_0} f|A(t)(t-z)^w dt + a_0(f|A) \int_0^z (t-z)^w dt.$$

Using the same relation for $\tilde{f}(AS^{-1})(Sz)j(S, z)^w$ with Sz_0 in place of z_0 we obtain, after a change of variables $u = St$ in the first integral above

$$\begin{aligned} (\tilde{f}|1-S)(A)(z) = & H_{z_0}(A)(z) - H_{Sz_0}(AS^{-1})|S(z) + \\ & + a_0(f|A) \int_0^z (t-z)^w dt - a_0(f|AS^{-1}) \int_0^{Sz} (t-Sz)^w j(S, z)^w dt. \end{aligned}$$

Computing the last integrals and comparing with (8.4) yields the conclusion. \square

We now determine the exact relationship between \widehat{W}_w^Γ and W_w^Γ . For $\widehat{P} \in \widehat{W}_w^\Gamma$ write $\widehat{P} = P + P_0$ where $P \in V_w^\Gamma$ and $P_0(A) = c_A X^{w+1} + d_A X^{-1}$ for $A \in \Gamma \setminus \Gamma_1$. From $\widehat{P}|1+S=0$ we obtain $d_A = c_{AS}$. From $\widehat{P}|1+U+U^2=0$, it follows that $P_0|1+U+U^2 \in V_w^\Gamma$, which implies that $c_A = c_{AT}$ for all $A \in \Gamma_1$. Therefore we have $P_0 = P^0|1-S$, where $P^0(A) = c_A X^{w+1}$, with $c_A = c_{AT}$. In conclusion, letting

$$(8.5) \quad D_w^\Gamma = \{(c_A X^{w+1})_A | (1-S) : c_A = c_{AT} = (-1)^w c_{AJ} \in \mathbb{C}\} \subset \widehat{V}_w^\Gamma$$

we have a unique decomposition of $\widehat{P} \in \widehat{W}_w^\Gamma$ as above

$$(8.6) \quad \widehat{P} = P + P^0|1-S, \quad P \in V_w^\Gamma, \quad P^0|1-S \in D_w^\Gamma.$$

For $\widehat{\rho}_f$ this is the decomposition in Proposition 8.1, since $a_0(f|A) = a_0(f|AT) = (-1)^w a_0(f|AJ)$. As in the proof of Lemma 4.2, note that $\dim D_w^\Gamma$ equals $e_\infty(\Gamma)$ or $e_\infty^{\text{reg}}(\Gamma)$, depending on whether k is even or odd respectively.

When $k = 2$ there is an extra relation satisfied by the coefficients of P^0 in (8.6). Letting $P(A) = d_A \in \mathbb{C}$, $P^0(A) = c_A X^{w+1}$, the period relations now imply that

$$d_A + d_{AU} + d_{AU^2} + 2(c_A + c_{AU} + c_{AU^2}) = 0, \text{ with } d_A + d_{AS} = 0, c_A = c_{AT}$$

for all $A \in \overline{\Gamma} \setminus \overline{\Gamma}_1$.⁶ Letting $e(A) = 2c_A + d_A$, and eliminating d_A using the relations above we find $4c_{ATS} = e(ATS) + e(ST^{-1}S) + e(AST^{-1}) + e(AT) - e(AS) - e(A)$. Summing over a system of representatives $A \in \overline{\Gamma} \setminus \overline{\Gamma}_1$ we obtain $2 \sum_A c_A = \sum_A e(A) = 0$, using $e(A) + e(AU) + e(AU^2) = 0$. We conclude that the coefficients c_A of P^0 satisfy $\sum_A c_A = 0$. From Proposition 8.1 it follows that $\sum_A a_0(f|A) = 0$ for all $f \in M_2(\Gamma)$.

Proposition 8.2. *a) If $k \geq 3$ there is an exact sequence*

$$0 \rightarrow W_w^\Gamma \rightarrow \widehat{W}_w^\Gamma \rightarrow D_w^\Gamma \rightarrow 0$$

where the first map is inclusion, and the second is the map $\widehat{P} \rightarrow P^0|1-S$ defined above.

b) If $k = 2$ there is an exact sequence

$$0 \rightarrow W_w^\Gamma \rightarrow \widehat{W}_w^\Gamma \rightarrow D_w^\Gamma \rightarrow \mathbb{C} \rightarrow 0$$

where the last map takes $P^0|1-S \in D_w^\Gamma$ with $P^0(A) = c_A X^{w+1}$ to $\sum_{A \in \Gamma \setminus \Gamma_1} c_A$.

⁶Since $\widehat{P}(A) = \widehat{P}(-A)$ when k is even, we can view c_A, d_A as being indexed by $A \in \overline{\Gamma} \setminus \overline{\Gamma}_1$.

Proof. Exactness at \widehat{W}_w^Γ follows from the definition. If $k \geq 3$, surjectivity of the second map follows from Proposition 8.1, and the fact that there is a basis of Eisenstein series $E_k^\mathcal{C} \in M_k(\Gamma)$ for $\mathcal{C} = [A_\mathcal{C}]$ a complete system of representatives for the (regular if k is odd) cusps in $\Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, such that $a_0(E_k^\mathcal{C}|A) = (-1)^w a_0(E_k^\mathcal{C}|AJ)$ equals 1 if $[A]^+ = [A_\mathcal{C}]^+$, and 0 if $[A] \neq \mathcal{C}$ (see Definition 4.3 for notation). If $k = 2$ the Eisenstein subspace of $M_2(\Gamma)$ is spanned by modular forms f which are nonzero only at a fixed pair of nonequivalent cusps and are zero at other cusps, and such that $\sum_{A \in \Gamma \backslash \Gamma_1} a_0(f|A) = 0$. Their images in D_w^Γ span the kernel of the last map, proving exactness at D_w^Γ . \square

The previous proposition shows that $\dim \widehat{W}_w^\Gamma = 2 \dim M_k(\Gamma)$. From the proof we conclude that there is a direct sum decomposition

$$(8.7) \quad \widehat{W}_w^\Gamma = W_w^\Gamma \oplus \widehat{E}_w(\Gamma)$$

where $\widehat{E}_w(\Gamma)$ is the image of the Eisenstein subspace $\mathcal{E}_k(\Gamma) \subset M_k(\Gamma)$ under the map $f \rightarrow \widehat{\rho}_f$.

The pairing $\{\cdot, \cdot\}$ extends to a pairing on $\widehat{W}_w^\Gamma \times \widehat{W}_w^\Gamma$, by decomposing $\widehat{P}, \widehat{Q} \in \widehat{W}_w^\Gamma$ as in (8.6) and setting:

$$(8.8) \quad \{\widehat{P}, \widehat{Q}\} = \langle\langle P|T - T^{-1}, Q \rangle\rangle + \langle\langle 2P^0|T - T^{-1}, Q \rangle\rangle + \langle\langle P, 2Q^0|T^{-1} - T \rangle\rangle + I_k(P^0, Q^0),$$

where $I_k(P^0, Q^0) = 0$ if k even and $I_k(P^0, Q^0) = \frac{6(k-1)}{k[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1} c_A c'_A$ if k is odd, where $P^0(A) = c_A X^{w+1}$, $Q^0(A) = c'_A X^{w+1}$.

Since $P^0|T - T^{-1} \in V_w^\Gamma$ this pairing is well-defined, and it is easily checked that it behaves as in (3.6) under the action of ϵ defined as in (2.2). We will show below that this definition is natural for two reasons: Haberland's formula generalizes to arbitrary modular forms, if the Petersson product is extended in a natural way to all modular forms, and this pairing is Hecke equivariant for $\Gamma = \Gamma_0(N)$, with the same action of Hecke operators on \widehat{W}_w^Γ as on W_w^Γ .

Recall that on W_w^Γ the pairing $\{\cdot, \cdot\}$ is degenerate, its radical being C_w^Γ (Lemma 4.1). We now show that the extended pairing is nondegenerate on \widehat{W}_w^Γ , more precisely that the dual of C_w^Γ inside \widehat{W}_w^Γ is the space $\widehat{E}_w(\Gamma)$.

Lemma 8.3. *a) Let $P = P'|1 - S \in C_w^\Gamma$ and $\widehat{Q} = Q + Q^0|1 - S \in \widehat{W}_w^\Gamma$, and let $P'(A) = c'_A$, $Q^0(A) = (-1)^w c_A \frac{X^{w+1}}{w+1}$ for $A \in \Gamma \backslash \Gamma_1$ (so that $c'_A = c'_{AT} = (-1)^w c'_{AJ}$, $c_A = c_{AT} = (-1)^w c_{AJ}$). Then*

$$\{P, \widehat{Q}\} = -\frac{6}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1} c'_A c_A.$$

b) The pairing $\{\cdot, \cdot\}$ is nondegenerate on \widehat{W}_w^Γ .

Proof. a) As in the proof of Lemma 4.1, we use the formal relation (4.1), together with the Γ_1 invariance of the pairing $\langle\langle \cdot, \cdot \rangle\rangle$, and the relation $(1 - S)(1 + U + U^2) = (1 - T^{-1})(1 + U + U^2)$:

$$\begin{aligned} \{P, \widehat{Q}\} &= \langle\langle P'|(1 - S)(T - T^{-1}), Q \rangle\rangle + 2 \langle\langle P', Q^0|(T^{-1} - T)(1 - S) \rangle\rangle \\ &= 2 \langle\langle P', Q|1 + U + U^2 \rangle\rangle + 2 \langle\langle P', Q^0|(T^{-1} - T)(1 - S) \rangle\rangle \\ &= -2 \langle\langle P', Q^0|[(1 - T^{-1})(1 + U + U^2) + (T - T^{-1})(1 - S)] \rangle\rangle \\ &= -2 \langle\langle P', Q^0|[2(1 - T^{-1}) + T - 1] \rangle\rangle \\ &= -\frac{6}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \sum_A c'_A c_A. \end{aligned}$$

b) We choose a basis of \widehat{W}_w^Γ by concatenating bases for $C_w^\Gamma, \rho^-(S_k(\Gamma)), \rho^+(S_k(\Gamma)), \widehat{E}_w^\Gamma$ in this order. The block form matrix of the pairing $\{\cdot, \cdot\}$ with respect to this basis is

$$\begin{pmatrix} 0 & 0 & 0 & \mathcal{A} \\ 0 & 0 & B & 0 \\ 0 & -B^t & 0 & 0 \\ -\mathcal{A}^t & 0 & 0 & C \end{pmatrix},$$

so it is enough to show that \mathcal{A} is nonsingular (B is nonsingular by Theorem 3.3). When $k > 2$ this is obvious from part a). For $k = 2$, we fix a cusp \mathcal{C}_0 in $\Gamma \backslash \Gamma_1 / \Gamma_{1\infty} = \{\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_n\}$ and we let a basis of C_w^Γ consist of P_i , $1 \leq i \leq n$, as in the statement of the lemma, with the constants $c'_{iA} = 1$ if $[A] \in \mathcal{C}_i$, and $c'_{iA} = 0$ otherwise (note that $P_0 = -\sum_{i=1}^n P_i$). Letting $l_i = \#\{A \in \overline{\Gamma} \backslash \overline{\Gamma}_1 : [A] \in \mathcal{C}_i\}$ (the width of the cusp \mathcal{C}_i), we take a basis of \widehat{E}_w^Γ to consist of \widehat{Q}_i as in the statement, $1 \leq i \leq n$, with $c_{iA} = 1$ if $[A] \in \mathcal{C}_i$, $c_{iA} = -\frac{l_i}{l_0}$ if $[A] \in \mathcal{C}_0$, and $c_{iA} = 0$ otherwise. With respect to this basis the matrix \mathcal{A} is diagonal, so the pairing is nondegenerate. \square

Assume now that Γ is normalized by ϵ . Since the action of ϵ given by (2.2) preserves \widehat{W}_w^Γ and D_w^Γ , passing to the ± 1 eigenspaces in Proposition 8.2 gives exact sequences

$$(8.9) \quad 0 \rightarrow (W_w^\Gamma)^\pm \rightarrow (\widehat{W}_w^\Gamma)^\pm \rightarrow (D_w^\Gamma)^\pm \rightarrow \mathbb{C} \rightarrow 0,$$

where the last map is nontrivial only if $k = 2$ and the sign is minus, when it is defined in Proposition 8.2 b) (when $k = 2$ and $P^0|1 - S \in (D_w^\Gamma)^+$ with $P^0(A) = c_A x^{w+1}$, then $c_A = -c_{A'}$ and $\sum_A c_A = 0$ automatically). From (4.2) and (8.5) we see that $\dim(D_w^\Gamma)^+ = \dim(C_w^\Gamma)^-$ for all k ; $\dim(D_w^\Gamma)^- = \dim(C_w^\Gamma)^+$ for $k \geq 3$; and $\dim(D_w^\Gamma)^- - 1 = \dim(C_w^\Gamma)^+$ for $k = 2$. Combined with the Eichler-Shimura isomorphism (2.5) and Lemma 4.2, this implies that $\dim(\widehat{W}_w^\Gamma)^\pm = \dim M_k(\Gamma)$.

The next Proposition can be seen as an extension of the Eichler-Shimura isomorphism (2.5) to the entire space of modular forms.

Proposition 8.4. *a) Assume that k, Γ are such that the extended Petersson scalar product on $M_k(\Gamma)$ defined in §8.2 is nondegenerate (see Remark 8.5). Then the maps $\widehat{\rho}^\pm : M_k(\Gamma) \rightarrow (\widehat{W}_w^\Gamma)^\pm$, $f \mapsto \widehat{\rho}_f^\pm$, are isomorphisms.*

b) Assume that $(C_w^\Gamma)^- = 0$ (for example $\Gamma = \Gamma_0(N)$ with N as in Proposition 4.4). Then $\widehat{\rho}^-$ is an isomorphism.

Remark 8.5. It is shown in [PP12] that the extended Petersson product is nondegenerate for $\Gamma_1(N)$ (and therefore also for $\Gamma_0(N)$) when $k > 2$. When $k = 2$ the extended Petersson product is nondegenerate for $\Gamma_1(p)$ or $\Gamma_0(p)$ with p prime, while it is degenerate for $\Gamma_0(N)$ with N squarefree with at least two prime factors. This implies that in the latter case the map $\widehat{\rho}^+$ is not an isomorphism; indeed, part b) shows that $\widehat{\rho}^-$ is an isomorphism, and if both $\widehat{\rho}^\pm$ were isomorphisms, then

the Petersson product would be nondegenerate by Theorem 8.6 c) below, since the pairing $\{\cdot, \cdot\}$ is nondegenerate.

Proof. a) Since the dimensions of the spaces are equal, we only have to prove injectivity. If $\widehat{\rho}_f^\pm = 0$, it follows from Theorem 8.6 c) that $f = 0$, when the extended Petersson product on $M_k(\Gamma)$ is nondegenerate.

b) If $(C_w^\Gamma)^- = 0$, then $(D_w^\Gamma)^+ = 0$ as well. Assuming $\widehat{\rho}_f^- = 0$ for $f \in M_k(\Gamma)$, it follows that $\rho_f^0 = 0$ so f is a cusp form, hence $f = 0$ by Theorem 3.3. \square

8.1. An example. As an example, we check directly that the map $\widehat{\rho}^+$ is not an isomorphism for $k = 2$ and $\Gamma = \Gamma_0(6)$. As explained in Remark 8.5, this gives an alternative proof that the extended Petersson product is degenerate in this case, in agreement with [PP12].

As representatives A_j for $\Gamma \backslash \Gamma_1$ we take the matrices

$$ST^{-i}S\{I, U^2, U\}, \quad i = 0, 1, 2, 3$$

in this order (namely $(A_1, A_2, \dots, A_{12}) = (I, U^2, U, ST^{-1}S, ST^{-1}SU^2, \dots, ST^{-3}SU)$), obtained from the set of representatives provided by the command ‘CosetRepresentatives’ in MAGMA. There are four cusps $\mathcal{C}_i \in \Gamma \backslash \Gamma_1 / \Gamma_{1\infty}$, and we have $\mathcal{C}_1 = [A_1]$, $\mathcal{C}_2 = [A_9] = [A_{12}]$, $\mathcal{C}_3 = [A_6] = [A_7] = [A_{11}]$, while the remaining six matrices are in the class \mathcal{C}_4 .

Since there are no cusp forms of weight two on $\Gamma_0(6)$, we have $W_w^\Gamma = C_w^\Gamma$. The latter space is spanned by polynomials P_i supported at the class \mathcal{C}_i , $i = 1, \dots, 3$, namely $P_i = (c_A)_A |1 - S$ with $c_A = c_{AT}$ and $c_A = 1$ if $[A] = \mathcal{C}_i$, $c_A = 0$ otherwise. We identify a polynomial $P \in C_w^\Gamma$ with a vector $\mathbf{d} = (d_i) \in \mathbb{C}^{12}$ with $P(A_i) = d_i$. Let $\sigma \in \mathcal{S}_{12}$ be the permutation such that $A_j S = A_{\sigma j}$. We have $\sigma = (3, 4, 1, 2, 7, 10, 5, 12, 11, 6, 9, 8)$, and it follows that the vectors \mathbf{d} corresponding to the polynomials P_1, P_2, P_3 are given respectively by (the entries not specified are equal to 0):

$$d_1 = 1, d_3 = -1; \quad d_9 = d_{12} = 1, d_{11} = d_8 = -1; \quad d_6 = d_7 = d_{11} = 1, d_{10} = d_5 = d_9 = -1.$$

Therefore in order to decompose a polynomial $P \in C_w^\Gamma$ with respect to the basis $\{P_1, P_2, P_3\}$ it is enough to know d_1, d_{12} and d_9 .

The space $M_2(\Gamma_0(6))$ is spanned by the Eisenstein series $E_2^t(z) = E_2(z) - tE_2(tz)$, for $t = 2, 3, 6$, where $E_2(z) = -\frac{1}{24} + \sum_{n \geq 1} \sigma_1(n) e^{2\pi i n z}$. Since $(D_w^\Gamma)^+ = (C_w^\Gamma)^- = 0$, we have $\widehat{\rho}^+(E_2^t) = \rho^+(E_2^t) \in C_w^\Gamma$. Letting $\rho(E_2^t)(A_i) = e_i$, $\rho(E_2^t)(A'_i) = e'_i$, we have $\rho^+(E_2^t)(A_i) = \frac{e_i + e'_i}{2} = d_i$, where $e'_j = e_{\tau j}$ with $\tau = (1, 4, 3, 2, 10, 7, 6, 8, 9, 5, 11, 12) \in \mathcal{S}_{12}$.

We now determine the constants d_i for each of the three Eisenstein series. Taking into account that $L(s, E_2) = \zeta(s)\zeta(s-1)$ and $L(s, E_2^t) = \zeta(s)\zeta(s-1)(1 - \frac{1}{ts-1})$, the constant d_1 is given by (8.2):

$$d_1 = C \ln(t), \quad \text{where } C = -\frac{\zeta(0)}{2\pi i}.$$

Since $E_2^2 \in M_2(\Gamma_0(2))$, and $A_6, A_7, A_{11} \in \Gamma_0(2)$ it follows that $e_1 = e_6 = e_7 = e_{11}$. We also have $e'_6 = e_7$, $e'_{11} = e_{11}$, hence $d_1 = d_6 = d_7 = d_{11}$. We obtain $\rho^+(E_2^2) = C \ln(2)(P_1 + P_3)$.

Similarly $E_2^3 \in M_2(\Gamma_0(3))$, and $A_9, A_{12} \in \Gamma_0(3)$, so $d_1 = d_9 = d_{12}$ and $\rho^+(E_2^3) = C \ln(3)(P_1 + P_2)$.

For E_2^6 , in order to determine d_9, d_{12} we need to find $L(s, E_2^6|A_i)$ for $i = 9, 12$. Note that $A_9, A_{12} \in \Gamma_0(3)$, so writing $E_2^6 = E_2^3 + 3E_2^2|(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix})$, it follows that

$$E_2^6|A_9 = E_2^3 + 3(E_2^2|S)|(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}), \quad E_2^6|A_{12} = E_2^3 + 3(E_2^2|ST)|(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}).$$

From the transformation properties of E_2 under Γ_1 we have $E_2|S(z) = E_2(z) - \frac{1}{2}E_2(\frac{z}{2})$, so $E_2^6|A_9(z) = E_2(z) - \frac{3}{2}E_2(\frac{3z}{2})$ and $E_2^6|A_{12} = E_2(z) - \frac{3}{2}E_2(\frac{3z+1}{2})$, obtaining

$$L(s, E_2^6|A_9) = \zeta(s)\zeta(s-1)\left(1 - \frac{2^{s-1}}{3^{s-1}}\right), \quad L(s, E_2^6|A_{12}) = \zeta(s)\zeta(s-1)\left(1 - \frac{2^{s-1}}{3^{s-1}}(3 \cdot 2^{1-s} - 1 - 2^{2-2s})\right)$$

and we find from (8.2) that $d_9 = C(\ln(3) - \ln(2))$, $d_{12} = C \ln 3$, so that

$$\rho^+(E_2^6) = C(P_1 \ln 6 + P_2 \ln 3 + P_3 \ln 2) = \rho^+(E_2^2) + \rho^+(E_2^3)$$

concluding that $\widehat{\rho}^+$ is not surjective.

8.2. Haberland's formula for arbitrary modular forms. The Petersson scalar product of two Eisenstein series of full level is defined by Zagier in [Za81]. Let \mathcal{F} be the fundamental domain $\{z \in \mathcal{H} : |z| \geq 1, |\operatorname{Re} z| \leq 1/2\}$ for Γ_1 , and for $T > 1$ let \mathcal{F}_T be the truncated domain for which $\operatorname{Im} z < T$. Since $\sum_A f|A(z)\overline{g|A(z)}y^k$ is a Γ_1 -invariant, renormalizable function in the sense of [Za81], we can define for $f, g \in M_k(\Gamma)$

$$(8.10) \quad (f, g) = \frac{1}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \lim_{T \rightarrow \infty} \sum_A \left[\int_{\mathcal{F}_T} f|A(z)\overline{g|A(z)}y^w dx dy - \frac{T^{k-1}}{k-1} a_0(f|A)\overline{a_0(g|A)} \right]$$

where the sum is over a complete system of representatives $A \in \overline{\Gamma} \backslash \overline{\Gamma}_1$. As in [Za81], it can be shown that the extended Petersson product equals $\operatorname{Res}_{s=k} L(s, f, \overline{g})$ up to a nonzero constant, where $L(s, f, \overline{g}) = \sum_{n \geq 1} \frac{a_n(f)\overline{a_n(g)}}{n^s}$. Using this fact, we show in [PP12] that for $\Gamma = \Gamma_1(N)$ the extended Petersson product is nondegenerate when $k > 2$, while for $k = 2$ it may or may not be degenerate.

We have the following generalization of Theorems 3.2 and 3.3.

Theorem 8.6. *Assume $k \geq 2$, and Γ is a finite index subgroup of Γ_1 . Let $f, g \in M_k(\Gamma)$.*

- a) *We have: $6C_k \cdot (f, g) = \{\widehat{\rho}_f, \widehat{\rho}_g\}$, where $C_k = -(2i)^{k-1}$.*
- b) *We have: $\{\widehat{\rho}_f, \widehat{\rho}_g\} = 0$.*
- c) *Assuming further that Γ is normalized by ϵ , and letting $\kappa_1, \kappa_2 \in \{+, -\}$ as in Theorem 3.3:*

$$3C_k \cdot (f, g) = \{\widehat{\rho}_f^{\kappa_1}, \widehat{\rho}_g^{\kappa_2}\}.$$

Proof. a) If one of f, g is a cusp form, then we can apply Stokes' theorem over the fundamental domain \mathcal{D} for $\Gamma(2)$ as in the proof of Theorem 3.2, and easily obtain the desired identity. When both f, g have nonzero constant terms, this approach is complicated by the fact that both f and g blow up at the cusps $-1, 0, 1$, and we prefer to apply Stokes' theorem to the domain \mathcal{F} as in [KZ84]. We use the following abbreviations: $f_A = f|A$, $g_A = g|A$, $a_A = a_0(f|A)$, $b_A = a_0(g|A)$, $C_\Gamma = [\overline{\Gamma}_1 : \overline{\Gamma}]$, $C_k = -(2i)^{k-1}$. Sums over A are over systems of representatives $A \in \overline{\Gamma} \backslash \overline{\Gamma}_1$. For all $T > 1$ we have

$$C_k C_\Gamma (f, g) = \sum_A \int_{\mathcal{F}} [f_A(z)\overline{g_A(z)} - a_A \overline{b_A}](z - \overline{z})^w dz d\overline{z} + a_A \overline{b_A} \left[\int_{\mathcal{F}_T} (z - \overline{z})^w dz d\overline{z} - C_k \frac{T^{k-1}}{k-1} \right]$$

By Stokes' we find $\int_{\mathcal{F}_T} (z - \overline{z})^w dz d\overline{z} = C_k \frac{T^{w+1}}{w+1} + \frac{1}{w+1} \int_{\rho^2}^\rho (z - \overline{z})^{w+1} d\overline{z}$. In the first integral we apply Stokes' theorem after writing $f_A \overline{g_A} - a_A \overline{b_A} = (f_A - a_A) \overline{g_A} + a_A (\overline{g_A} - \overline{b_A})$ to get

$$C_k C_\Gamma (f, g) = \sum_A \int_{\partial \mathcal{F}} -F_A(z)\overline{g_A(z)} + a_A [\overline{g_A(z)} - \overline{b_A}] \frac{(z - \overline{z})^{w+1}}{w+1} d\overline{z} + \frac{a_A \overline{b_A}}{w+1} \int_{\rho^2}^\rho (z - \overline{z})^{w+1} d\overline{z}$$

where $F_A(z) = \int_z^{i\infty} [f_A(t) - a_A](t - \bar{z})^w dt$. Since $F_{AT}(z) = F_A(Tz)$, the integrals over the vertical sides of \mathcal{F} cancel (after summing over A) and setting $\tilde{F}_A(z) = F_A(z) - a_A \int_0^z (t - \bar{z})^w dt$ we obtain:

$$C_k C_\Gamma(f, g) = \sum_A \int_\rho^{\rho^2} \tilde{F}_A(z) \overline{g_A}(z) d\bar{z} + (-1)^w \frac{a_A}{w+1} \int_\rho^{\rho^2} \overline{g_A}(z) \bar{z}^{w+1} d\bar{z}.$$

In the first integral we change variables $z \rightarrow Sz$, which reverses the order of integration. As in the proof of Proposition 8.1 we have $\tilde{F}_A(z) - \tilde{F}_{AS^{-1}}|_{-w} S(z) = \rho_f(A)(\bar{z})$ obtaining

$$(8.11) \quad C_k C_\Gamma(f, g) = \sum_A \frac{1}{2} \int_\rho^{\rho^2} \rho_f(A)(\bar{z}) \overline{g_A}(z) d\bar{z} + (-1)^w \frac{a_A}{w+1} \int_\rho^{\rho^2} \overline{g_A}(z) \bar{z}^{w+1} d\bar{z}.$$

We now proceed to write the result in terms of the pairing $\langle \cdot, \cdot \rangle$ on V_w^Γ . Define $H_{z_0} \in V_w^\Gamma$ as in (8.3), with g in place of f . Using $\int_\rho^{\rho^2} \overline{g_A}(z)(\bar{z} - X)^w d\bar{z} = \overline{H}_\rho(A) - \overline{H}_{\rho^2}(A)$ and (3.1) we get

$$\int_\rho^{\rho^2} \rho_f(A)(\bar{z}) \overline{g_A}(z) d\bar{z} = \left\langle \rho_f(A), \int_\rho^{\rho^2} \overline{g_A}(z)(\bar{z} - X)^w d\bar{z} \right\rangle = \langle \rho_f(A), \overline{H}_\rho(A) - \overline{H}_{\rho^2}(A) \rangle.$$

The second integral in (8.11) can be written

$$\int_\rho^{\rho^2} \overline{g_A}(z) \bar{z}^{w+1} d\bar{z} = \overline{b_A} \int_\rho^{\rho^2} \bar{z}^{w+1} d\bar{z} + \int_\rho^{i\infty} - \int_{\rho^2}^{i\infty} (\overline{g_A}(z) - \overline{b_A}) \bar{z}^{w+1} d\bar{z}$$

and changing variables $z = t - 1$ in the last integral, recalling that $\rho_f^0(A) = (-1)^w a_A \frac{X^{w+1}}{w+1}$ ($= \rho_f^0(AT)$), and using (3.1) we obtain

$$\begin{aligned} \sum_A (-1)^w \frac{a_A}{w+1} \int_\rho^{\rho^2} \overline{g_A}(z) \bar{z}^{w+1} d\bar{z} &= \sum_A \langle \rho_f^0(A) | 1 - T^{-1}, \overline{H}_\rho(A) \rangle + \\ &\quad \sum_A a_A \overline{b_A} \left[\int_0^1 \int_0^\rho (t - \bar{z})^w d\bar{z} dt + (-1)^w \int_\rho^{\rho^2} \frac{\bar{z}^{w+1}}{w+1} d\bar{z} \right]. \end{aligned}$$

The expression inside square brackets equals $\frac{1}{(k-1)k}$, and setting $I(f, g) = \frac{2}{k(k-1)C_\Gamma} \sum_A a_A \overline{b_A}$ we get

$$2C_k(f, g) = \langle \rho_f, \overline{H}_\rho - \overline{H}_{\rho^2} \rangle + 2 \langle \rho_f^0 | 1 - T^{-1}, \overline{H}_\rho \rangle + I(f, g).$$

Now we use

$$(8.12) \quad H_{\rho^2} = H_\rho |T - \rho_g^0| (1 - T), \quad \rho_g = H_\rho - H_{\rho^2} |S = H_\rho | (1 - TS) + \rho_g^0 | (1 - T) S.$$

Taking into account the relation $(1 + U^2) = \frac{1}{3}(U^2 - U)(1 - U^{-1}) + \frac{2}{3}(1 + U + U^2)$ in $\mathbb{Q}[\overline{\Gamma}_1]$ we have

$$\begin{aligned} \langle \rho_f, \overline{H}_\rho | 1 - T \rangle &= \langle \rho_f | 1 - T^{-1}, \overline{H}_\rho \rangle = \langle \rho_f | 1 + ST^{-1}, \overline{H}_\rho \rangle = \\ &= \frac{1}{3} \langle \rho_f | U^2 - U, \overline{H}_\rho | 1 - U \rangle + \frac{2}{3} \langle \rho_f | 1 + U + U^2, \overline{H}_\rho \rangle \\ &= \frac{1}{3} \langle \rho_f | -T^{-1} - TS, \overline{\rho_g} - \overline{\rho_g^0} | (1 - T) S \rangle + \frac{2}{3} \langle \rho_f | 1 + U + U^2, \overline{H}_\rho \rangle \\ &= \frac{1}{3} \langle \rho_f | T - T^{-1}, \overline{\rho_g} \rangle + \frac{1}{3} \langle \rho_f | T + T^{-1} S, \overline{\rho_g^0} | 1 - T \rangle - \\ &\quad - \frac{2}{3} \langle \rho_f^0 | (1 - T^{-1})(1 + U + U^2), \overline{H}_\rho \rangle \end{aligned}$$

(on the last line we used Proposition 8.1) and collecting terms we get

$$6C_k(f, g) = \langle \rho_f | T - T^{-1}, \overline{\rho_g} \rangle + \langle \rho_f | 3 + T^{-1}S + T, \overline{\rho_g^0} | 1 - T \rangle + \\ + 2 \langle \rho_f^0 | (1 - T^{-1})(2 - U - U^2), \overline{H_\rho} \rangle + 3I(f, g)$$

In the second term we use the relation

$$(1 - T)(3 + ST + T^{-1}) = 2(T^{-1} - T) + (1 - T)S[1 + U + U^2 - U^2(1 + S)]S,$$

while in the third we use $2 - U - U^2 = (1 - U)(1 - U^2)$ and (8.12):

$$(8.13) \quad 6C_k(f, g) = \langle \rho_f | T - T^{-1}, \overline{\rho_g} \rangle + \langle \rho_f, 2\overline{\rho_g^0} | (T^{-1} - T) \rangle + \langle 2\rho_f^0 | (T - T^{-1}), \overline{\rho_g} \rangle + \\ + \langle \rho_f^0 | (1 - T)(T^{-1}S - ST - 3), \overline{\rho_g^0} | 1 - T \rangle + 3I(f, g)$$

Let $p(X) = \frac{X^{w+1}}{w+1}|1 - T$, $q(X) = \frac{X^{w+1}}{w+1}|1 - T^{-1}$. Then

$$\langle \rho_f^0 | (1 - T)(T^{-1}S - ST), \overline{\rho_g^0} | 1 - T \rangle = \frac{1}{C_\Gamma} \sum_A (a_A \overline{b_{AS^{-1}}} - (-1)^w a_{AS^{-1}} \overline{b_A}) \langle p, q | S \rangle = 0,$$

where we changed A to AS^{-1} in one of the sums and used that $a_{AJ} = (-1)^w a_A$. We also have

$$\langle \rho_f^0 | (1 - T), \overline{\rho_g^0} | 1 - T \rangle = \frac{1}{C_\Gamma} \sum_A a_A \overline{b_A} \langle p, p \rangle = \frac{1 + (-1)^w}{k(k-1)C_\Gamma} \sum_A a_A \overline{b_A},$$

which vanishes if w is odd, and equals $I(f, g)$ if w is even. Therefore the second line in (8.13) vanishes if k is even, and it equals $3I(f, g) = I_k(\rho_f^0, \rho_g^0)$ if k is odd, finishing the proof.

b) Going backwards in the proof of part a) up to the first equation after applying Stokes' theorem, we obtain

$$C_\Gamma \{\widehat{\rho}_f, \widehat{\rho}_g\} = -6 \sum_A \int_{\partial \mathcal{F}} \widetilde{f}(A)(z) g_A(z) dz$$

with \widetilde{f} defined in (8.1). Since the integrand is holomorphic and vanishes at $i\infty$, each term vanishes.

c) Since the extended pairing $\{\cdot, \cdot\}$ behaves as the original one under the action of ϵ , the claim follows from a) and b) as in the proof of Theorem 3.3. \square

8.3. Hecke operators. We now pass to the action of Hecke operators on \widehat{W}_w^Γ . We define an action of the Hecke operators \widetilde{T}_n on \widehat{W}_w^Γ as in Section 5. Although matrices in the definition of \widetilde{T}_n do not preserve \widehat{V}_w^Γ , we have the following generalization of Proposition 5.1 and of Corollary 5.2.

Proposition 8.7. a) With the hypotheses of Proposition 5.1, we have $\widehat{\rho}_f|_{T_n} = \widehat{\rho}_f|\widetilde{T}_n$ for $f \in M_k(\Gamma)$.

b) With the hypothesis of Corollary 5.2, we have $\widehat{\rho}_f^\pm|_{T_n} = \widehat{\rho}_f^\pm|\widetilde{T}_n$ for $f \in M_k(\Gamma)$.

c) The operators \widetilde{T}_n preserve the space \widehat{W}_w^Γ .

Proof. a) The proof is the same as of Proposition 5.1, once we show that $\widetilde{f}|_{T_n}^\infty = \widetilde{f}|\widetilde{T}_n$. Equation (5.3) becomes

$$\begin{aligned} \widetilde{f}|_{T_n}^\infty(A) &= \sum_{M \in M_n^\infty \cap \Gamma_1 \Delta_n A} \int_{Mz}^{i\infty} [f|A_M(t) - a_0(f|A_M)](t - Mz)^w j(M, z)^w dt \\ &= n^{w+1} \sum_{M \in M_n^\infty \cap \Gamma_1 \Delta_n A} \int_z^{i\infty} [f|M_A A(u) - a_0(f|A_M)] j(M, u)^{-k} (u - z)^w du \end{aligned}$$

As in Proposition 5.1, we obtain $\widetilde{f}|T_n^\infty(A) = \int_z^{i\infty} [(f|T_n)|A - c(n, f, A)](u - z)^w du$ where $c(n, f, A)$ is the sum of the terms involving $a_0(f|A_M)$ (which is independent of u since $j(M, u) = d_M$). Since the integral converges, we must have $c(n, f, A) = a_0(f|T_n|A)$, hence the last expression equals $\widetilde{f}|T_n(A)$ finishing the proof.

b) The proof is the same as of Corollary 5.2.

c) This follows from part a) and the decomposition (8.7) □

Proposition 8.8. *Assume $\Gamma = \Gamma_0(N)$ and n coprime to N . We have for all $\widehat{P}, \widehat{Q} \in \widehat{W}_w^\Gamma$*

$$\{\widehat{P}|\widetilde{T}_n, \widehat{Q}\} = \{\widehat{P}, \widehat{Q}|\widetilde{T}_n\}.$$

Proof. The claim follows as in the first proof of Theorem 5.5, using Theorem 8.6, Proposition 8.7 and the Hecke equivariance of the extended Petersson inner product on $M_k(\Gamma)$ [PP12]. □

The next corollary shows that the trace of Hecke operators T_n on the Eisenstein subspace $\mathcal{E}_k(\Gamma)$ is the same as the trace of \widetilde{T}_n on C_w^Γ . For Γ_1 , when $C_w^{\Gamma_1} = \langle X^w - 1 \rangle$, a direct proof is immediate, but for $\Gamma_0(N)$ it seems difficult to prove the statement without using the dual space \widehat{E}_w^Γ .

Corollary 8.9. *For $\Gamma = \Gamma_0(N)$ we have:*

$$\mathrm{Tr}(T_n, \mathcal{E}_k(\Gamma)) = \mathrm{Tr}(\widetilde{T}_n, \widehat{E}_w^\Gamma) = \mathrm{Tr}(\widetilde{T}_n, C_w^\Gamma).$$

Proof. The first equality follows from Proposition 8.7 a), and the second from Proposition 8.8, and the fact that \widehat{E}_w^Γ and C_w^Γ are dual under the nondegenerate pairing $\{\cdot, \cdot\}$ (see the proof of Lemma 8.3). □

Remark 8.10. The corollary shows that $\mathrm{Tr}(\widetilde{T}_n, W_w^\Gamma) = \mathrm{Tr}(T_n, M_k(\Gamma)) + \mathrm{Tr}(T_n, S_k(\Gamma))$. For $\Gamma = \Gamma_1$, this was used by Zagier to sketch an elementary proof of the Eichler-Selberg trace formula for Γ_1 , by computing directly the left side for an appropriately chosen \widetilde{T}_n [Za93]. It would be interesting to generalize this proof of the trace formula to $\Gamma_0(N)$.

8.4. Extra relations revisited. Theorem 8.6 gives another way of determining the extra relations satisfied by all period polynomials of cusp forms which are independent of the period relations. Assuming that $\widehat{\rho}^-$ is an isomorphism (see Proposition 8.4), it follows that there exist $g \in \mathcal{E}_k(\Gamma)$ such that $\widehat{\rho}_g^-$ form a basis for $(\widehat{E}_w^\Gamma)^-$. Since the pairing $\{\cdot, \cdot\}$ is nondegenerate, it follows that the linear relations $\{P, \widehat{\rho}_g^-\} = 0$ are satisfied by $P = \rho_f^+$, for all $f \in S_k(\Gamma)$, but they are not satisfied by some $P \in (C_w^\Gamma)^+$. A similar argument applies to determine the relations satisfied by ρ_f^- , when $(C_w^\Gamma)^- \neq 0$ and $\widehat{\rho}^+$ is an isomorphism. These linear relations can be used to define other versions of the linear forms λ_+, λ_- in Proposition 7.1, which are entirely explicit once the period polynomials of Eisenstein series are determined.

As an example we take $\Gamma = \Gamma_1(N)$, and we assume $k \geq 3$. Then the Eisenstein subspace $\mathcal{E}_k(\Gamma)$ has a basis of Eisenstein series which are Hecke eigenforms for the Hecke operators of index coprime with the level [DS05, Ch. 5]. Their period polynomials for the identity coset can be determined in terms of special values of Dirichlet L -functions by Proposition 8.1. For other cosets $A \in \Gamma \backslash \Gamma_1$ the period polynomials of the Hecke eigenforms are harder to compute. Instead, consider a second basis, consisting of Eisenstein series which vanish at all but one cusp, so that the action $|A$ permutes the elements of this basis. The elements of the second basis can be decomposed in terms of Hecke eigenforms, so their period polynomials corresponding to all cosets A can be determined explicitly. We omit the tedious details.

9. ALGEBRAIC PROPERTIES OF HECKE OPERATORS

In this section we prove Theorem 5.6:

$$(9.1) \quad \tilde{T}_n(T - T^{-1}) + (T^{-1} - T)\tilde{T}_n^\vee \in \mathcal{I} + \mathcal{I}^\vee, \quad \mathcal{I} = (1 + S)R_n + (1 + U + U^2)R_n,$$

along the way obtaining other algebraic properties of the operators $\tilde{T}_n, T_n^\infty \in R_n$. In §9.1 we reduce (9.1) to relations involving only T_n^∞ , and possessing an extra symmetry besides invariance under taking adjoint. The main part of the proof is contained in §9.2, where we also derive the indefinite theta series identities from the introduction as corollaries.

9.1. Preliminary reductions. First we prove a characterization of the set $\mathcal{I} + \mathcal{I}^\vee$, based on a similar characterization of \mathcal{I} in [CZ93].

Proposition 9.1. *Let $A \in R_n = \mathbb{Q}[M_n]$. Then $A \in \mathcal{I} + \mathcal{I}^\vee$ if and only if*

$$(1 - S)A(1 - S) \in (1 - T)R_n(1 - S) + (1 - S)R_n(1 - T^{-1}).$$

Proof. The proof is based on the characterization of \mathcal{I} in [CZ93][Lemma 3]:

$$(9.2) \quad v \in \mathcal{I} \iff (1 - S)v \in (1 - T)R_n.$$

We need a more precise version, which appears in the proof of Lemma 3 in [CZ93]:

Lemma 9.2 ([CZ93]). *Let $A, B \in R_n$ such that $(1 - S)A = (1 - T)B$. Then there exists $C \in R_n$ such that*

$$A = (1 + S)C - SB, \text{ with } SB \in (1 + U + U^2)R_n.$$

If $A \in \mathcal{I} + \mathcal{I}^\vee$, the claim of Proposition 9.1 follows immediately from (9.2) and its adjoint. Assume therefore that $A \in R_n$ satisfies:

$$(1 - S)A(1 - S) = (1 - T)\alpha(1 - S) + (1 - S)\beta(1 - T^{-1})$$

By the adjoint of the relations in Lemma 9.2 it follows that there exists $C \in R_n$ such that

$$(9.3) \quad (1 - S)A = (1 - T)\alpha + C(1 + S) + (1 - S)\beta S,$$

with $(1 - S)\beta S \in R_n(1 + U + U^2)$. Since $(1 \pm S)^2 = 2(1 \pm S)$, multiplying (9.3) by 2 we get

$$(1 - S) \cdot 2A = (1 - T) \cdot 2\alpha + C(1 + S)(1 + S) + (1 - S)(1 - S)\beta S$$

From (9.3) we can write $C(1 + S) = (1 - S)\gamma + (1 - T)\delta$, therefore:

$$(1 - S)[2A - \gamma(1 + S) - (1 - S)\beta S] \in (1 - T)R_n.$$

By (9.2) we conclude

$$2A - \gamma(1 + S) - (1 - S)\beta S \in \mathcal{I}.$$

Since $(1 - S)\beta S \in R_n(1 + U + U^2)$, it follows by dividing the last equation by 2 that $A \in \mathcal{I} + \mathcal{I}^\vee$. \square

Corollary 9.3. *Let $\mathcal{J} = (1 - T)R_n(1 - S) \subset R_n$. Then (9.1) is equivalent to the statement*

$$(9.4) \quad T_n^\infty S T^{-1}(1 - S) + (1 - S)T S T_n^{\infty \vee} \in \mathcal{J} + \mathcal{J}^\vee.$$

We remark that $T_n^\infty A(1 - S) \pmod{\mathcal{J}}$ is well-defined modulo multiplication by powers of T on the left, for any $A \in \Gamma_1$, so we can take in (9.4)

$$(9.5) \quad T_n^\infty = \sum_{n=ad} \sum_{b \pmod{d}} t_{a,d}(b), \quad \text{where } t_{a,d}(b) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Proof. By Proposition 9.1 and (5.1), (9.1) is equivalent to the following statement:

$$T_n^\infty(1-S)(T-T^{-1})(1-S) + (1-S)(T^{-1}-T)(1-S)T_n^{\infty\vee} \in \mathcal{J} + \mathcal{J}^\vee.$$

By the remark above, we have $T_n^\infty(1-T)(1-S) \in \mathcal{J}$, so that:

$$\begin{aligned} T_n^\infty(1-S)(T-T^{-1})(1-S) &\equiv T_n^\infty S(T^{-1}-T)(1-S) \\ &\equiv T_n^\infty(ST^{-1} + T^{-1}ST^{-1})(1-S) \\ &\equiv 2T_n^\infty ST^{-1}(1-S) \pmod{\mathcal{J}} \end{aligned}$$

On the second line we used $STS = T^{-1}ST^{-1}$, and on the third the remark above. \square

We need the following notation. Let $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and for $m \in M_n$ denote $m' = \epsilon m \epsilon$. For any matrix $m \in M_n$ we denote by $m^* \in R_n$ the following linear combination of eight matrices:

$$(9.6) \quad m^* = (m - m')(1-S) + (1-S)(m^\vee - m'^\vee)$$

and extend this notation to elements of R_n by linearity. Note that $m'^\vee = m^{\vee'}$.

In the next proposition we show that (9.4) is equivalent with more symmetric relations, involving only terms of type m^* . The latter will be proved in the next section, using only the following easily checked properties:

$$(P1) \quad m^* + (m'^\vee)^* + (SmS)^* + (Sm'^\vee S)^* = 0$$

$$(P2) \quad (m')^* = -m^*$$

$$(P3) \quad (mS)^* = -m^*$$

$$(P4) \quad (T^k m)^* \equiv m^* \pmod{\mathcal{J} + \mathcal{J}^\vee} \text{ for any integer } k.$$

Proposition 9.4. *We have the following congruences $\pmod{\mathcal{J} + \mathcal{J}^\vee}$:*

$$(9.7) \quad \sum_{n=ad} \left[\sum_{-d-\frac{a}{2} < k < d-\frac{a}{2}} \begin{pmatrix} a+k & -k \\ -d & d \end{pmatrix} - \delta\left(\frac{a}{2}\right) \begin{pmatrix} d & -d \\ a/2 & a/2 \end{pmatrix} \right]^* \equiv 4 [T_n^\infty U^2(1-S) + (1-S)UT_n^{\infty\vee}]$$

$$(9.8) \quad \sum_{n=ad} \left[2 \sum_{-\frac{a}{2} < k < \frac{d-a}{2}} \begin{pmatrix} d & -d \\ -k & a+k \end{pmatrix} + \delta\left(\frac{a}{2}\right) \begin{pmatrix} d & -d \\ a/2 & a/2 \end{pmatrix} \right]^* \equiv 2 [T_n^\infty U^2(1-S) + (1-S)UT_n^{\infty\vee}],$$

where $\delta(x)$ is 1 if $x \in \mathbb{Z}$ and 0 otherwise.

The LHS of (9.7) is congruent to $2(T_n^\infty U^2)^*$, while the LHS of (9.8) is congruent with $(UT_n^{\infty\vee} U^2)^*$, but we will not need these facts in the sequel.

Proof. The summand in the LHS of (9.7) is $(t_{a,d}(-k-a)ST^{-1})^*$ [with $t_{a,d}(b)$ defined in (9.5)], and we note that k is well-defined modulo d by (P4). We can rewrite the LHS as follows:

$$\sum_{n=ad} \sum_{k \bmod d} 2(t_{a,d}(k)ST^{-1})^* - \delta\left(\frac{a}{2}\right) \left[\begin{pmatrix} a/2 & a/2 \\ -d & d \end{pmatrix} + \begin{pmatrix} d & -d \\ a/2 & a/2 \end{pmatrix} \right]^* \equiv 2(T_n^\infty ST^{-1})^*,$$

because the sum of the terms in square brackets equals

$$\sum_{n=2ad} \left[\begin{pmatrix} a & a \\ -d & d \end{pmatrix} + \begin{pmatrix} d & -d \\ a & a \end{pmatrix} \right]^* \equiv 0$$

(exchanging a with d in the first matrix and using (P2)).

For any $A \in R_n$, we use the abbreviation $\{A\} + \{\text{adj}\} := A + A^\vee$. We have

$$\begin{aligned}
 (T_n^\infty ST^{-1})^* &\equiv \{[T_n^\infty ST^{-1} - (T_n^\infty)'ST](1-S)\} + \{\text{adj}\} \\
 &\equiv \{[T_n^\infty ST^{-1} + (T_n^\infty)STS](1-S)\} + \{\text{adj}\} \\
 &\equiv \{2T_n^\infty ST^{-1}(1-S)\} + \{\text{adj}\},
 \end{aligned}
 \tag{9.9}$$

proving (9.7). On the second line we have used that $(T_n^\infty)'A(1-S) \equiv T_n^\infty A(1-S)$ for any $A \in \Gamma_1(\mathbb{Z})$, by the remark following Corollary 9.3, while on the third we used $(ST)^3 = 1$ together with the fact that $T_n^\infty T^{-1}(1-S) \equiv T_n^\infty(1-S) \pmod{\mathcal{J}}$, by the same remark.

The summand in (9.8) is $(St_{a,d}(k)ST^{-1})^*$, and we have as in (9.9):

$$(St_{a,d}(k)ST^{-1})^* = \{(St_{a,d}(k)ST^{-1} + St_{a,d}(-k-a)ST^{-1})(1-S)\} + \{\text{adj}\}.$$

The range $-\frac{a+d}{2} < k < \frac{d-a}{2}$ is invariant under $k \rightarrow -k-a$, hence the LHS of (9.8), which we denote L_n , becomes:

$$L_n \equiv \sum_{n=ad} \left\{ 2 \sum_{-\frac{a+d}{2} < k < \frac{d-a}{2}} St_{a,d}(k)ST^{-1}(1-S) \right\} + \{\text{adj}\}.$$

Now we have:

$$St_{a,d}(k)ST^{-1}(1-S) \equiv t_{a,d}(k)ST^{-1}(1-S) + (1-S)[t_{a,d}(a+k) - t_{a,d}(k)]S.$$

Since $t_{a,d}(k)ST^{-1}(1-S) \pmod{\mathcal{J}}$ depends only on $k \pmod{d}$ we have:

$$\sum_{-\frac{a+d}{2} < k < \frac{d-a}{2}} t_{a,d}(k)U^2(1-S) \equiv \sum_{k=0}^{d-1} t_{a,d}(k)U^2(1-S) - \delta\left(\frac{d-a}{2}\right) t_{a,d}\left(\frac{d-a}{2}\right)U^2(1-S).$$

Similarly

$$\sum_{-\frac{a+d}{2} < k < \frac{d-a}{2}} [t_{a,d}(a+k) - t_{a,d}(k)](S-1) \equiv \delta\left(\frac{d-a}{2}\right) [t_{a,d}\left(\frac{d-a}{2}\right) - t_{a,d}\left(\frac{a-d}{2}\right)](S-1)$$

Next we observe that

$$\begin{aligned}
 \sum_{n=ad} \sum_{-\frac{a+d}{2} < k < \frac{d-a}{2}} \{t_{a,d}(k) - t_{a,d}(a+k)\} + \{\text{adj}\} &\equiv \sum_{n=ad} \left[\sum_{-\frac{a+d}{2} < k < \frac{d-a}{2}} [t_{a,d}(k) - t_{a,d}(a+k)] + \right. \\
 &\quad \left. + \sum_{-\frac{a+d}{2} < k < \frac{a-d}{2}} [t_{a,d}(-k) - t_{a,d}(-d-k)] \right] \equiv \sum_{n=ad} \delta\left(\frac{d-a}{2}\right) [t_{a,d}\left(\frac{a-d}{2}\right) - t_{a,d}\left(\frac{d-a}{2}\right)].
 \end{aligned}
 \tag{9.13}$$

Notice that (9.13) can be conjugated by S , since $S^\vee = S$. Putting together the resulting equation, and (9.13), (9.12), (9.11), (9.10), we obtain:

$$L_n \equiv \{2T_n^\infty ST^{-1}(1-S)\} + \{\text{adj}\} + 2 \sum_{n=ad} \delta\left(\frac{d-a}{2}\right) E_{a,d}$$

where:

$$E_{a,d} = S \left[t_{a,d} \left(\frac{a-d}{2} \right) - t_{a,d} \left(\frac{d-a}{2} \right) \right] S + t_{a,d} \left(\frac{d-a}{2} \right) - t_{a,d} \left(\frac{a-d}{2} \right) + \\ + \left\{ \left[t_{a,d} \left(\frac{d-a}{2} \right) - t_{a,d} \left(\frac{a-d}{2} \right) \right] (S-1) - t_{a,d} \left(\frac{d-a}{2} \right) U^2 (1-S) \right\} + \{\text{adj}\}.$$

It is easy to see that $E_{a,d} + E_{d,a} \equiv 0 \pmod{\mathcal{J} + \mathcal{J}^\vee}$, finishing the proof of (9.8). \square

9.2. Main algebraic result and a theta series identity. In this section, we finish the proof of (9.1). The indefinite theta series identities from the introduction follow as a bonus from the proof, and since they are of independent interest we keep this section self-contained.

Recall the notation (9.6) and properties (P1)-(P4), which are the only ingredients needed. For convenience we rewrite the four-term relation:

$$(P1) \quad \begin{pmatrix} a & -b \\ c & d \end{pmatrix}^* + \begin{pmatrix} d & -b \\ c & a \end{pmatrix}^* + \begin{pmatrix} a & -c \\ b & d \end{pmatrix}^* + \begin{pmatrix} d & -c \\ b & a \end{pmatrix}^* = 0.$$

We will apply this relation to matrices in the following set

$$\mathcal{S}_n = \left\{ \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in R_n : c \geq b; \quad d \geq a; \quad a+b > d-c > 0 \right\}.$$

More precisely, for each $\gamma = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}$, denote $A_\gamma = \left\{ \begin{pmatrix} a & -b \\ c & d \end{pmatrix}, \begin{pmatrix} d & -b \\ c & a \end{pmatrix}, \begin{pmatrix} a & -c \\ b & d \end{pmatrix}, \begin{pmatrix} d & -c \\ b & a \end{pmatrix} \right\}$ and define (the union is disjoint)

$$(9.14) \quad \mathcal{T}_n := \bigcup_{\gamma \in \mathcal{S}_n} A_\gamma = \left\{ \begin{pmatrix} a & -b \\ c & d \end{pmatrix} : b+c > |a-d|, \quad \max(a,d) > \max(b,c) \right\}$$

so that we have by construction

$$(9.15) \quad \sum_{\gamma \in \mathcal{T}_n} \gamma^* = 0.$$

It is convenient to further symmetrize the set \mathcal{T}_n by interchanging $a \leftrightarrow b$, $c \leftrightarrow d$. Note that if a, b, c, d satisfy $\max(a,d) \geq \max(b,c)$ then

$$b+c > |a-d| \iff c+d > |a-b|, \quad a+b > |c-d|.$$

Defining therefore

$$(9.16) \quad \begin{aligned} \mathcal{U}_n &:= \mathcal{T}_n \cup \left\{ \begin{pmatrix} b & -a \\ d & c \end{pmatrix} : \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{T}_n \right\} \\ \mathcal{X}_n &:= \left\{ \begin{pmatrix} a & -b \\ c & d \end{pmatrix} : c+d > |a-b|, \quad a+b > |c-d| \right\} \\ \mathcal{V}_n &:= \left\{ \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{X}_n : \max(a,d) = \max(b,c) \right\}, \end{aligned}$$

we have the disjoint union $\mathcal{X}_n = \mathcal{U}_n \cup \mathcal{V}_n$. By (P2), (P3), (9.15) we have $\sum_{\gamma \in \mathcal{U}_n} \gamma^* = 0$, and it is easily found that

$$(9.17) \quad \mathcal{V}_n = \bigcup_{\gamma} A_\gamma \quad \text{over } \gamma = \begin{pmatrix} a & -b \\ c & c \end{pmatrix} \text{ with } c|n, \quad c \geq a \geq \frac{n}{c} - c$$

so that $\sum_{\gamma \in \mathcal{V}_n} \gamma^* = 0$. Consequently

$$(9.18) \quad \sum_{\gamma \in \mathcal{X}_n} \gamma^* = 0.$$

Proposition 9.5. *For every $n > 0$ we have the congruence (mod $\mathcal{J} + \mathcal{J}^\vee$):*

$$(9.19) \quad \sum_{\gamma \in \mathcal{X}_n} \gamma^* \equiv \sum_{n=ad} \left[\sum_{-d-\frac{a}{2} < k < d-\frac{a}{2}} \binom{a+k}{d} \binom{k}{d} + 2 \sum_{-\frac{a}{2} < k < \frac{d-a}{2}} \binom{d}{k} \binom{d}{a+k} \right]^*.$$

By Proposition 9.4 and (P2), we obtain

$$0 = \sum_{\gamma \in \mathcal{X}_n} \gamma^* \equiv -6 [T_n^\infty U^2(1-S) + (1-S)UT_n^{\infty\vee}] \pmod{\mathcal{J} + \mathcal{J}^\vee},$$

which, together with Corollary 9.3, finishes the proof of Theorem 5.6.

Before proving the proposition, we show that it implies the theta series identity (1.1). For a set of 2 by 2 matrices A denote by $\mathcal{N}(A)$ the number of matrices $\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in A$ with $cd > 0$, minus the number of matrices with $cd < 0$, namely:

$$\mathcal{N}(A) = \sum_{\gamma \in A} \text{sgn}(c_\gamma d_\gamma).$$

Corollary 9.6. *We have the identity:*

$$(9.20) \quad \sum'_{\substack{a,b,c,d \geq 0 \\ a+b > |d-c|, \ c+d > |a-b|}} q^{ad+bc} = 3\tilde{E}_2(q) - 2 \sum_{n>0} \sigma_{\min}(n) q^n + \sum_{n>0} q^{n^2},$$

where $\sigma_{\min}(n) = \sum_{n=ad} \min(a, d)$.

Proof. We claim that $\mathcal{N}(\mathcal{X}_n)$ equals the coefficient of q^n in the LHS of (9.20); on the other hand, going from (9.18) to (9.19) we use only relations (P2)-(P4), which do not change the function \mathcal{N} , so $\mathcal{N}(\mathcal{X}_n)$ can be computed exactly from the RHS of (9.19), yielding the coefficient of q^n in the RHS of (9.20), and the proof is finished.

To count $\mathcal{N}(\mathcal{X}_n)$, we start with \mathcal{T}_n . Let $\gamma = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{S}_n$. Clearly $c, d > 0$ and we have three cases:

- (1) $a > 0 > b$ or $b > 0 > a$. Then $\mathcal{N}(A_\gamma) = 0$;
- (2) $a, b > 0$. Then $\mathcal{N}(A_\gamma) = |A_\gamma|$;
- (3) One of a, b is 0. Then the other is positive and $\mathcal{N}(A_\gamma) = \frac{1}{2}|A_\gamma|$.

By (9.14) we conclude that $\mathcal{N}(\mathcal{T}_n) = \#\left\{ \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{T}_n : a, b, c, d \geq 0 \right\}$, each matrix with $abcd = 0$ being counted with weight 1/2. The same is obviously true for \mathcal{U}_n , and by inspection for \mathcal{V}_n (see (9.17)). We conclude that

$$(9.21) \quad \mathcal{N}(\mathcal{X}_n) = \# \left\{ \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{X}_n : a, b, c, d \geq 0; \begin{array}{l} \text{matrices with } abcd = 0 \text{ are} \\ \text{counted with weight } 1/2 \end{array} \right\},$$

that is $\mathcal{N}(\mathcal{X}_n)$ is the coefficient of q^n in the LHS of (9.20), as claimed. \square

Proof of Proposition 9.5. We decompose \mathcal{X}_n as a disjoint union:

$$\mathcal{X}_n = \mathcal{X}_n^< \cup \mathcal{X}_n^= \cup \mathcal{X}_n^>$$

where $\mathcal{X}_n^<$, $\mathcal{X}_n^=$, and $\mathcal{X}_n^>$ consist of those $\gamma = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{X}_n$ with $|c| < d$, $c = d$, and $c > |d|$ respectively. Using (P2)-(P4), we will show that $\sum_{\gamma \in \mathcal{X}_n} \gamma^*$ reduces to (9.19).

If $\begin{pmatrix} x & k \\ d & d \end{pmatrix} \in \mathcal{X}_n^=$, we have $n = ad$, $x = a + k$ and $2d > |a + 2k|$, hence

$$(9.22) \quad \sum_{\gamma \in \mathcal{X}_n^=} \gamma^* = \sum_{n=ad} \sum_{-d-\frac{a}{2} < k < d-\frac{a}{2}} \binom{a+k}{d} \binom{k}{d}^*.$$

Since $m \rightarrow m'S$ takes $\mathcal{X}_n^<$ bijectively onto $\mathcal{X}_n^>$, we have by (P2), (P3)

$$(9.23) \quad \sum_{\gamma \in \mathcal{X}_n^<} \gamma^* = \sum_{\gamma \in \mathcal{X}_n^>} \gamma^*.$$

It remains to calculate $\sum_{\gamma \in \mathcal{X}_n^<} \gamma^*$.

For $m \in M_n$, denote by $\{m\}$ the equivalence class of m in $\mathcal{X}_n^<$ modulo multiplication by powers of T on the left, namely $\{m\} := \Gamma_{1\infty} m \cap \mathcal{X}_n^<$ where $\Gamma_{1\infty} = \{T^k : k \in \mathbb{Z}\}$. Consider the involution

$$f : \Gamma_{1\infty} \backslash \mathcal{X}_n^< \rightarrow \Gamma_{1\infty} \backslash \mathcal{X}_n^<, \quad f(\{m\}) = \{m'\}.$$

We show that f is bijective, and the classes $\{m\}$, $m \in \mathcal{X}_n$ have one or two elements. Let $m = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{X}_n^<$, and let $T^{-k}m' = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}$. Then

$$T^{-k}m' \in \mathcal{X}_n \iff k > 1 + \frac{b-a}{c+d}, \quad \left| k - \frac{a+b}{d-c} \right| < 1.$$

Since $c+d > |b-a|$, $a+b > d-c > 0$, it is clear that there exist one or two values of k for which $T^{-k}m' \in \mathcal{X}_n$, and therefore f is bijective (since it is an involution). Moreover,

$$(9.24) \quad |\{m'\}| = \begin{cases} 1 & \text{if } d-c|a+b \quad (k = \frac{a+b}{d-c}); \\ 1 & \text{if } d-c \nmid a+b, \quad b \geq a, \quad \frac{a+b}{d-c} < 2 \quad (k=2); \\ 2 & \text{otherwise.} \end{cases}$$

Since $d-c|a+b \iff A=B$, and $a=b \iff D-C|A+B$, and $b > a, \frac{a+b}{d-c} < 2 \iff B > A, \frac{A+B}{D-C} < 2$, there are four possibilities:

- (1) $d-c \nmid a+b, a \neq b \iff D-C \nmid A+B, A \neq B$. Then $|\{m\}| = |\{m'\}|$.
- (2) $d-c|a+b, a=b \iff D-C|A+B, A=B$. Then $|\{m\}| = |\{m'\}| = 1$.
- (3) $d-c \nmid a+b, a=b \iff D-C|A+B, A \neq B$. Then

$$|\{m\}| = 1, \quad |\{m'\}| = \begin{cases} 1 & \text{if } \frac{a+b}{d-c} < 2 \\ 2 & \text{if } \frac{a+b}{d-c} > 2 \end{cases}.$$

- (4) $d-c|a+b, a \neq b \iff D-C \nmid A+B, A=B$. Then

$$|\{m'\}| = 1, \quad |\{m\}| = \begin{cases} 1 & \text{if } \frac{A+B}{D-C} < 2 \\ 2 & \text{if } \frac{A+B}{D-C} > 2 \end{cases}.$$

Summing over two copies of $\mathcal{X}_n^<$ divided into classes $\{m\}$ and respectively $\{m'\}$, everything cancels by (P2) except for matrices m' in case (3) with $\frac{a+b}{d-c} > 2$, and matrices m in case (4) for which $\frac{A+B}{D-C} > 2$. The two sums are equal and we obtain

$$2 \sum_{\gamma \in \mathcal{X}_n^<} \gamma^* = 2 \sum_{\gamma} \gamma^*, \quad \text{over } \gamma = \begin{pmatrix} a & a \\ -c & d \end{pmatrix}, d > |c|, d-c \nmid 2a, \frac{2a}{d-c} > 2.$$

Writing $n = ar$, so that $c+d = r$, we obtain (after substituting $-c$ for c)

$$\sum_{\gamma \in \mathcal{X}_n^<} \gamma^* \equiv \sum_{n=ar} \sum_{\substack{0 < r+2c < a \\ r+2c \nmid 2a}} \begin{pmatrix} a & a \\ c & c+r \end{pmatrix}.$$

By (9.22), (9.23), the proof of (9.19) is finished once we show that the same sum, but with the congruence condition replaced by $r + 2c \mid 2a$, yields 0 (mod $\mathcal{J} + \mathcal{J}^\vee$). By (P2), (P4), it is enough to show that

$$\begin{pmatrix} a & a \\ c & c+r \end{pmatrix} = \begin{pmatrix} a'-tc' & -a'+t(c'+r') \\ -c' & c'+r' \end{pmatrix} \iff r + 2c \mid 2a, \quad r' + 2c' \mid 2a',$$

where $n = ar = a'r'$, $t \in \mathbb{Z}$, and $0 < r + 2c < a$, $0 < r' + 2c' < a'$. Indeed the equality of the matrices implies $t = \frac{2a}{r+2c} = \frac{2a'}{r'+2c'}$, proving the equivalence above. \square

We end this section with a proof of (1.2), which is entirely similar to that of Corollary 9.6. Let $\mathcal{S}'_n, \dots, \mathcal{X}'_n$ be the sets defined like $\mathcal{S}_n, \dots, \mathcal{X}_n$ but with the extra parity conditions $a \equiv d \pmod{2}$, $b \equiv c \pmod{2}$.

Proposition 9.7. *a) If n is odd then*

$$(9.25) \quad \sum_{\gamma \in \mathcal{X}'_n} \gamma^* \equiv \sum_{n=ad} 2 \sum_{-\frac{a}{2} < k < \frac{d-a}{2}} \begin{pmatrix} d & d \\ k & a+k \end{pmatrix}^*.$$

b) If n is even then

$$(9.26) \quad \sum_{\gamma \in \mathcal{X}'_n} \gamma^* \equiv \sum_{\substack{n=ad \\ 2 \mid a}} \sum_{\substack{-d-\frac{a}{2} < k < d-\frac{a}{2} \\ k \equiv d \pmod{2}}} \begin{pmatrix} a+k & k \\ d & d \end{pmatrix}^* + \sum_{\substack{n=ad \\ 2 \mid a, 2 \nmid d}} 2 \sum_{-\frac{a}{2} < k < \frac{d-a}{2}} \begin{pmatrix} d & d \\ k & a+k \end{pmatrix}^*.$$

Proof. The proof is very similar to that of Proposition 9.5, and we prove only part a) leaving b) as an exercise for the reader. Assuming n odd and referring to the proof of Proposition 9.5, the parity condition implies that $a \not\equiv b, c \not\equiv d \pmod{2}$, so that $\mathcal{X}'_n = \emptyset$, and only cases (1) and (4) can occur for $m \in \mathcal{X}'_n$. The integer k such that $T^{-k}m' \in \mathcal{X}'_n$ must be even, so in case (1) we have $|\{m\}| = |\{m'\}| = 1$, while in case (4) we have $|\{m'\}| = 0$, because $\frac{a+b}{d-c}$ is odd in that case.

Denoting by $\mathcal{X}'_n^{(4)}$ the set of $m = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \in \mathcal{X}'_n$, with $d > |c|, d - c \mid a + b$, it follows that

$$\sum_{\gamma \in \mathcal{X}'_n} \gamma^* \equiv \sum_{m \in \mathcal{X}'_n^{(4)}} m^*$$

Writing $d - c = r > 0, a + b = lr$, from $a(d - c) + c(a + b) = n$ we have that $n = rs$ with $s \in \mathbb{Z}$ and $a + lc = s$, so $T^l m = \begin{pmatrix} s & s \\ c & c+r \end{pmatrix}$. The condition $m \in \mathcal{X}'_n$ implies that

$$l > 1, l \text{ odd, and } \left| l - \frac{2s}{2c+r} \right| < 1,$$

so there is a unique such m for each $c > \frac{-r}{2}$ such that $\frac{2s}{2c+r} > 2$, and $2c + r \nmid 2s$. Consequently:

$$\sum_{\gamma \in \mathcal{X}'_n} \gamma^* \equiv \sum_{n=rs} \sum_{\substack{\frac{-r}{2} < c < \frac{s-r}{2} \\ 2c+r \nmid 2s}} \begin{pmatrix} s & s \\ c & c+r \end{pmatrix}^*.$$

The proof is finished by observing that the divisibility condition can be removed, exactly as in the previous case. \square

Corollary 9.8. *We have the identity*

$$(9.27) \quad \sum'_{\substack{x \geq |y|, z \geq |t| \\ x > |t|, z > |y|}} q^{x^2+z^2-y^2-t^2} = \tilde{E}_2(q) - 2\tilde{E}_2(q^2) + 4\tilde{E}_2(q^4) - 2 \sum_{n>0} \sigma_{\min}^{\text{ev}}(n) q^n + \sum_{n>0} q^{n^2},$$

where $\sigma_{\min}^{\text{ev}}(n) = \sum_{n=ad, 2|(d-a)} \min(a, d)$.

Proof. Arguing as before, we have that $\mathcal{N}(\mathcal{X}'_n)$ is given by (9.21) with \mathcal{X}_n replaced by \mathcal{X}'_n . Making a substitution $a = x + y, d = x - y, b = z + t, c = z - t$, the conditions $a, b, c, d \geq 0$ become $x \geq |y|, z \geq |t|$, while $a + b > |c - d|, c + d > |a - b|$ become $x > |t|, z > |y|$. Therefore $\mathcal{N}(\mathcal{X}'_n)$ is the coefficient of q^n in the LHS of (9.27).

As before, we can count $\mathcal{N}(\mathcal{X}'_n)$ from Proposition 9.7. When n is odd, (9.25) gives

$$\mathcal{N}(\mathcal{X}'_n) = \sigma(n) - 2\sigma_{\min}(n) + \delta(\sqrt{n}).$$

If n is even, (9.26) yields (the first term comes from the first sum, and the second from the second sum)

$$\mathcal{N}(\mathcal{X}'_n) = \left[\sigma\left(\frac{n}{2}\right) - \tau\left(\frac{n}{4}\right) \right] + \left[2\sigma\left(\frac{n}{4}\right) + \tau\left(\frac{n}{4}\right) - 2\sigma_{\min}^{\text{ev}}(n) + \delta\left(\frac{\sqrt{n}}{2}\right) \right],$$

where $\tau(n)$ is the number of divisors of n and we adopt the convention that an arithmetic function is zero on non-integers. This is exactly the coefficient of q^n in the RHS of (9.27) \square

The relations in Section 9 have been checked numerically for small n using MAGMA [Mgm].

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